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THEORETICAL BETA DECAY CALCULATIONS

EMPLOYING THE NILSSON MODEL

A THESIS

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EMPLOYING THE NILSSON MODEL

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SUMMARY

Nuclear beta decay is used as a means of either determining nuclear wave functions or testing nuclear models. Two particle Nilsson wave functions have been employed to calculate the nuclear matrix elements for the decay of ${}_{75}Re^{186}$ and ${}_{69}Tm^{170}$, two odd-odd nuclei that have similar decay schemes. It has been found[1] that they are in good agreement with experiment for the decay of Re^{186} but in poor agreement for the decay of Tm^{170}.

To determine whether this discrepancy is due to the particular wave functions used or to the radial approximations which have been utilized in the evaluation of these matrix elements, the beta decay observables are calculated without making use of the above approximations. To do this, expressions for the normalized shape factor and the beta-gamma angular correlation were developed. This was done using the V-A beta decay law under the assumptions that the nuclei could be treated non-relativistically and that the emitted electron sees the uniform charge distribution of the daughter nucleus.

The radial Dirac equation was solved numerically for the electron wave functions both for a uniform charge distribution and a Fermi charge distribution (Hofstadter potential). It appeared that the more realistic Fermi charge distribution would not significantly change the results; hence the rest of the analysis was carried out with the simpler uniform charge distribution.

Next, using two particle Nilsson wave functions, the radial integrals for various initial and final Nilsson parameters were integrated numerically and the observables obtained in this manner were compared (1) with experiment, (2) with the values obtained using the radial approximation, and (3) with the two term Buhring approximation. For the case of Re^{186} and Tm^{170}, the elimination of this approximation did not significantly effect the calculated observables. Hence the two particle Nilsson intrinsic wave functions used are not valid for Tm^{170} but do fit the experimental results for Re^{186} rather well.

Finally, in the last chapter, two methods are described that could be used to test the validity of the radial approximations for a particular calculation.

CHAPTER I

INTRODUCTION

When a radioactive nucleus ${}_{Z}X^{A}$ undergoes beta decay

$$ {}_{Z}X^{A} \rightarrow {}_{Z+1}Y^{A*} + e^{-} + \bar{\nu} $$

several observables can be measured in the laboratory. Two examples are, the probability that such a decay will occur and the probability that the electron (e^{-}) being emitted will have a momentum of magnitude p. If the daughter nucleus ${}_{Z+1}Y^{A}$ itself then decays by emitting a gamma ray

$$ {}_{Z+1}Y^{A*} \rightarrow {}_{Z+1}Y^{A} + \gamma $$

then the probability that the gamma ray is emitted at a certain angle with respect to the electron can be experimentally determined.

These experimental data depend upon the properties of the nuclei and the leptons (electrons and neutrinos) as well as the beta decay interaction.

The properties of the leptons are believed to be adequately described by Dirac's relativistic mechanics. Since 1956 and the discovery of non-conservation of parity, it is believed that the V-A beta decay law is adequate to describe the interaction when the momentum involved is relatively low.

The object of beta decay theory now is to develop a formulation, using the V-A law and Dirac theory, to utilize the experimental data to determine the properties of nuclei. This is done in the framework of quantum theory. First order perturbation theory is used since beta decay is a weak interaction.

To calculate any of the beta decay observables, it is necessary to evaluate the beta decay matrix elements

$$H_{Ii} = \int \psi_I^+ O_\mu \psi_i \psi_e^+ O_\mu \psi_\nu^c \, d\tau$$

After inserting the explicit forms of the well understood parts of this matrix element, namely the electron and neutrino wave functions, ψ_e and ψ_ν, as well as the Dirac operators, O_μ, this is in too general a form to use for the determination of the nuclear contribution to this interaction, ψ_I and ψ_i. The next step is to assume a model of the nucleus that is simple enough to be useful but is sufficiently general to incorporate all of the various known nuclear phenomena.

One model that is employed is the unified model. This incorporates the features of the liquid drop model, which explained fission in the late thirties, with the single particle model, which explained the magic numbers in the late forties. It exploits the observation that low lying nuclear energy levels for nuclei near the magic numbers have vibrational-like energy levels, while nuclei far from the magic numbers seem to have rotational-like energy levels. With this model, A. Bohr[2] envisioned the nucleus as being non-spherical, pulsating, and rotating.

The Nilsson model[3] which treats the nucleus as being cigar shaped and rotating should be most valid for nuclei far from the magic numbers.

Tuong[1] has used two particle Nilsson wave functions to evaluate Kotani parameters which then can be used in the Morita and Morita[4] formalism to calculate beta decay observables. This is done to evaluate the beta-gamma A_2 angular correlation coefficients for the decay of Tm^{170} and Re^{186}. These two nuclei have similar decay schemes.

It was found that, while the calculated values of the A_2 coefficient for Re^{186} fit rather well the experimentally observed values, the fit for Tm^{170} was rather poor. The question then arose whether the lack of fit for Tm^{170} was due to the model employed, or to the particular states used, or perhaps to approximations made in the course of the calculations.

It is possible that this error is due to cancellation of the nuclear matrix elements which could possibly introduce large errors in the Morita and Morita formalism.

Here, an investigation of the radial approximation will be made for the purpose of gaining a better understanding of nuclear beta decay.

Expressions for Some Beta Decay Observables

In this work the following beta decay observables will be calculated: the normalized shape correction factor and the beta-gamma A_2 angular correlation coefficient.

To obtain expressions for these observables, one can begin with the experimental fact that the number of electron emissions for a given sample decays exponentially with time, i.e.

$$N = N_0 e^{-\lambda t}$$

where λ is the probability per unit time that a transition occurs. Since the beta decay interaction is a weak interaction, first order perturbation theory should be sufficient to describe the beta decay. Using it, one arrives at Fermi's second golden rule that transition rate

$$\lambda = \frac{2\pi}{\hbar} \sum |H_{Ii}|^2$$

Since the decay is to a continuum of final electron energy states, this sum goes to an integral

$$\lambda = \frac{\ln 2}{t} = \frac{2\pi}{\hbar} \int \sum |H_{Ii}\ (E)|^2\ \rho(E)\ dE$$

where t is the half life, and $\rho(E)$ is the density of final states which depends on the normalization of the wave functions involved.

The probability that an electron of energy E is emitted per unit time, i.e., the electron spectrum shape is

$$\frac{2\pi}{\hbar} \sum |H_{Ii}(E)|^2\ \rho(E)$$

This should depend on three factors, one is the number of ways an electron of a given energy can be emitted, ρ_s, i.e., the statistical factor. Since the electron after being emitted from the nucleus is in the coulomb field of the daughter atom, the electron shape experimentally observed will have more electrons at lower momentum than those actually emitted by the nuclei. This is taken into account by the Fermi function, F. The final contribution to the electron shape will then be due to the beta decay interaction itself and is called the shape factor, S. Hence

$$\sum |H_{Ii}\ (E)|^2\ \rho(E) = S(E)\ F(Z,E)\ \rho_s(E)$$

Normalized Shape Factor

Since experimentally ratios are easier to measure than absolute values, a normalized shape factor is introduced, i.e.

$$N_s(E) \equiv \frac{S(E)}{S(E_R)}$$

where $S(E_R)$ is the shape at some reference energy. Hence the expression for the half life

$$\frac{1}{t} = \frac{2\pi}{\hbar\ \ln 2} \int_{m_e c^2}^{E_{max}} |H_{Ii}\ (E)|^2\ \rho(E)\ dE$$

can be written as

$$\frac{1}{t} = \frac{2\pi}{\hbar\ \ln 2}\ S(E_R) \int N_S(E)\ F(Z,W)\ \rho_s(E)\ dE$$

If the normalized shape factor is independent of energy, the so-called "allowed shape," then the strength of the beta decay interaction is given by

$$S = \frac{\bar{h}\ \ln 2}{2\pi}\frac{1}{ft}$$

where f is defined as

$$f = \int F(Z,W)\ \rho_s(E)\ dE$$

Expressions and values for the f's are given in the first chapter of reference 5.

Beta-Gamma Angular Correlation Coefficients

To get an expression for the beta-gamma A_2 coefficient, one begins with an expression first given by Hamilton.[6,7] The probability that a photon of polarization S is immediately emitted at an angle θ with respect to the emitted electron, i.e., the beta-gamma angular correlation function, $W(\theta,S)$ is given by

$$W(\theta,S) = \sum \left| \sum |H_{Ii}{}^{\beta}\ H_{fI}{}^{\gamma} \right|^2$$

As shown in Appendix III, this can be put into the following form

$$W(\theta,S) = \sum_{k=o}^{\infty} S^k\ A_k{}'\ P_k(\cos\ \theta)$$

The beta-gamma A_k angular correlation coefficients are then defined as

$$A_k = \frac{A_k{}'}{A_o{}'}$$

In this work, the A_2 coefficient will be calculated by evaluating the coefficient of the Legendre polynomial $P_2(\cos\theta)$ in the expression for the beta-gamma angular correlation function. The log ft value can be determined from the expression

$$\frac{1}{ft} = \frac{2\pi}{\hbar \, f \, \ln 2} \int \sum |H_{Ii}\ (E)|^2 \ \rho(E) \ dE$$

The normalized shape factor will be determined from

$$N_s(E) = \frac{\Sigma |H_{Ii}\ (E)|^2 \ \rho(E)}{F(Z,E)\ \rho_s(E)} \Bigg/ \frac{\Sigma |H_{Ii}\ (E_R)|^2 \ \rho(E_R)}{F(Z,E_R)\ \rho_s(E_R)}$$

Hence to evaluate these as well as all other observables, the beta decay matrix element must be evaluated, i.e.

$$H_{Ii} = \int \psi_I^{+} \ H \ \psi_i \ d\tau = \int h \ d\tau$$

where ψ_i is the nuclear wave function before the beta transition, ψ_I is the nuclear wave function after transition, H is the beta decay interaction Hamiltonian, and h is the Hamiltonian density.

The V-A Law, Non-Relativistic in Nucleons

The beta decay interaction will be assumed to be the V-A law as given by Konopinski.[5,8] As shown in Appendix I, this can be written as

$$h = \frac{g}{\sqrt{2}} \{ \psi_I^{+} (C_V \vec{\alpha} + C_A \vec{\sigma}) \psi_i \cdot \vec{A}$$

$$+ \psi_I^{+} (C_V - C_A \gamma_5) \psi_i A_4 \}$$

where A_μ, the lepton term, is defined as

$$\vec{A} = \psi_e^{+} (\pm Z) \vec{\sigma} (\pm 1 + \gamma_5) \psi_\nu^{c}$$

$$A_4 = \psi_e^{+} (\pm Z) (1 \pm \gamma_5) \psi_\nu^{c}$$

ψ_e and ψ_ν are the electron and neutrino wave functions, Z is the charge of the daughter nucleus, the upper sign is for negatron emission, the lower for positron emission, and $\vec{\alpha}$, $\vec{\sigma}$, and γ_5 are the Dirac matrices. The strength of the interaction is given by

$$gCv = 2.87(10^{-12})$$

which is in natural units, i.e., $c = \hbar = m_e = 1$, and

$$\frac{C_A}{C_V} = -1.19$$

Next, in Appendix I, the approximation is made that the nucleons can be described by non-relativistic wave functions.

$$\psi = \begin{pmatrix} 1 \\ \frac{c\sigma \cdot p}{W - V + mc^2} \end{pmatrix} U \rightarrow \begin{pmatrix} 1 \\ \frac{\sigma \cdot p}{2mc} \end{pmatrix} U$$

Since $|V| \ll mc^2$ and $|p| \ll mc$.

Using this and keeping terms only up to order $1/m$ because the $1/m^2$ terms would give contributions in first order perturbation theory of the same magnitude as the $1/m$ terms would in second order perturbation theory, one gets the results of Rose and Osburn[7] who used a Foldy-Wouthuysen transformation.

$$\psi_I^+ \vec{\alpha}\, \psi_i \cdot A \rightarrow U_I^+ \frac{A\cdot p}{Mc} U_i + \frac{1}{2Mc}\left(U_I^+ U_i\, p\cdot A + i\, U_I^+ \vec{\sigma}\, U_i \cdot p \times A\right)$$

$$\psi_I^+ \vec{\sigma}\, \psi_i \cdot A \rightarrow U_I^+ \vec{\sigma}\, U_i \cdot A$$

$$\psi_I^+ \psi_i\, A_4 \rightarrow U_I^+ U_i\, A_4$$

$$\psi_I^+ \gamma_5\, \psi_i A_4 \rightarrow -\, A_4\, U_I^+ \frac{\sigma\cdot p}{Mc} U_i - \frac{U_I\, \vec{\sigma}\, U_i \cdot p\, A_4}{2Mc}$$

Hence the non-relativistic in nucleon beta decay Hamiltonian can be written as

$$\int \psi_I^+ H\, \psi_i\, d\tau = \frac{g}{\sqrt{2}} \int r^2 dr\, de \left\{ C_V U_I^+ U_i \left(A_4 + \frac{\vec{p}\cdot\vec{A}}{2Mc}\right) \right.$$

$$+ C_V U_I^+ \frac{\vec{A}\cdot\vec{p}}{Mc} U_i + C_A\, U_I^+ \frac{A_4\, \vec{\sigma}\cdot\vec{p}}{Mc} U_i$$

$$\left. + U_I^+ \vec{\sigma}\, U_i \cdot \left(C_A\, \vec{A} + \frac{C_A\, \vec{p}\, A_4}{2Mc} + i\, \frac{C_V\, \vec{p} \times \vec{A}}{2Mc}\right) \right\}$$

The Lepton Multipole Expansion for Spherical Potentials

Since the total angular momentum and parity seem to be good quantum numbers for nuclear states, the next appropriate step is to put the beta decay matrix element in a useful form. This is done by writing the lepton contribution A_μ in the spherical representation. This is done in Appendix II under the assumption that the emitted electron sees a central potential which goes to zero at large distances and that the neutrino is free, both of which satisfy the relativistic Dirac equation. This yields

$$h = \frac{g}{\sqrt{2}} \left(\frac{W}{p}\right)^{\frac{1}{2}} q \sum (-i)^L \, e^{i\Delta_\kappa} \, (-)^{\mu+\mu_\nu+\frac{1}{2}-j_\nu+\ell_\nu} \, D^{j\,*}_{\mu\sigma}(Z\to p) \, D^{j_\nu}_{\mu_\nu\sigma_\nu}(Z\to q)$$

$$\kappa,\ \kappa_\nu = \pm 1,\ \pm 2,\ ---$$

$$\mu = j,\ j-1, ---, -j$$

$$\mu_\nu = j,\ j-1, ---, -j$$

$$J = 0,\ 1,\ 2,\ ---$$

$$0 \leqq L = J,\ J\pm 1$$

$$((2\ell+1)(2\ell_\nu+1))^{\frac{1}{2}} \, (\ell\ \tfrac{1}{2}\ j:0\sigma\sigma)(\ell_\nu\ \tfrac{1}{2}\ j:0\sigma_\nu\sigma_\nu)(j j_\nu\ J:\mu\mu_\nu M) \ \rho_J(j\ j_\nu)$$

$$\{D_1(JJ\kappa\kappa_\nu)\ U_I^+\ Y_{J-M}\ U_i + D_2(JL\kappa\kappa_\nu)\ U_I^+\ \vec{\sigma}\cdot\vec{V}^{\,L}_{J-M}\ U_i$$

$$+ D_3(JL\kappa\kappa_\nu)\ U_I^+\ \vec{V}^{\,L}_{J-M}\cdot\vec{p}\ U_i + D_4(JJ\kappa\kappa_\nu)\ U_I^+\ Y_{J-M}\ \sigma\cdot p\ U_i\}$$

The symbols appearing above are defined as follows. The Dirac angular momentum quantum number

$$\kappa = \pm\ (j + \tfrac{1}{2}) \quad \text{for } j = \ell \pm \tfrac{1}{2}$$

j and ℓ are the usual total angular momentum and orbital angular momentum quantum numbers. The spin of the fermions is $\frac{1}{2}$, and μ, m, and σ are their Z quantum numbers, respectively. Δ_{κ} is the phase shift due to the electron's potential. $D^{j}_{\mu\sigma}$ are the rotation matrices. $(j_1\ j_2\ j_3 : \mu_1\ \mu_2\ \mu_3)$ are the Clebsch-Gordan coefficients. Y_{JM} are the spherical harmonics. $\vec{V}_{J\,M}^{\,L}$ are the "vector spherical harmonics" which Konopinski[5] calls $\vec{T}_{J\,M}^{\,L}$. Konopinski's notation is not adopted because of possible confusion with Rose and Osburn's[10] irreducible spherical tensors

$$T^{m}_{JL}\ (\hat{r},\vec{B}) = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\ (-)^{L+1-J}\ \vec{V}^{\,L}_{Jm}\cdot\vec{B}$$

The D's are defined as follows and contain the radial part of the lepton contribution.

$$D_1(JJ\kappa\kappa_\nu) = i^L\ C_V \left\{\delta_{LJ}\ A_4\ (JL\kappa\kappa_\nu)\ \frac{-i\,\hbar}{2mc}\left[\delta_{L,J-1}\left(\frac{2L+1}{2L+3}\right)^{\frac{1}{2}} D_-(L)\ A(JL\kappa\kappa_\nu)\right.\right.$$

$$\left.\left.+\ \delta_{L,J+1}\left(\frac{L}{2L-1}\right)^{\frac{1}{2}} D+(L)\ A(JL\kappa\kappa_\nu)\right]\right\}$$

$$\rho_J(jj_\nu) \quad = \left(\frac{(2j+1)(2j_\nu+1)}{4\pi\ (2J+1)}\right)^{\frac{1}{2}}\ (jj_\nu J\ :\ -\tfrac{1}{2}\tfrac{1}{2}\ 0)$$

$$D_2(JL\kappa\kappa_\nu) = i^L\left\{C_A\ A(JL\kappa\kappa_\nu) + \frac{i\,\hbar}{2mc}\left(\delta_{L,J+L}\ C_A\left(\frac{L}{2L-1}\right)^{\frac{1}{2}} D_-(J)A_4(JJ\kappa\kappa_\nu)\right.\right.$$

$$+\ \delta_{L,J-1}\ C_A\left(\frac{L+1}{2L+3}\right)^{\frac{1}{2}} D_+(J)A_4(JJ\kappa\kappa_\nu)$$

$$+\left[\frac{C_V}{2}\left(\frac{(L+J+2)(L-J+1)(L+J-1)(J-L+2)}{(2L-1)(2L+1)}\right)^{\frac{1}{2}}\right.$$

$$\left.D_-(L-1)A(J,L-1,\kappa\kappa_\nu)\right]_{L>0}$$

$$+ \frac{C_V}{2}\left(\frac{(L+J+3)(L+J)(L-J+2)(J-L+1)}{(2L+3)(2L+1)}\right)^{\frac{1}{2}}$$

$$D_+(L+1)\left(A(J,J+1,\kappa\kappa_\nu)\right)\Big\}$$

$$D_3(JL\kappa\kappa_\nu) = i^L \frac{C_V\ A(JL\kappa\kappa_\nu)}{mc}$$

$$D_4(JJ\kappa\kappa_\nu) = i^L \frac{\delta_{JL}\ C_A\ A_4(LL\kappa\kappa_\nu)}{mc}$$

The $D_\pm$ are defined as follows

$$D_-(L) = \frac{d}{dr} - \frac{L}{r} \qquad \text{and}$$

$$D_+(L) = \frac{d}{dr}\ \frac{L+1}{r} = D_-(1)\ \frac{2L+1}{r}$$

and the A's are given by

$$A_4(LL\kappa\kappa_\nu) = \pm\ \delta_{\ell+\ell_\nu+L,\text{even}}\ \overline{D(\kappa\kappa_\nu)} + i\ \frac{\kappa_\nu}{|\kappa_\nu|}\ \delta_{\ell+\bar{\ell}_\nu+L,\text{even}}\ \overline{D(\kappa-\kappa_\nu)}$$

$$A(JL\kappa\kappa_\nu) = \delta_{\ell+\ell_\nu+L,\text{even}}\ D_{JL}(\kappa\kappa_\nu) \pm i\ \frac{\kappa_\nu}{|\kappa_\nu|}\ \delta_{\ell+\bar{\ell}_\nu+L,\text{even}}\ D_{JL}(\kappa-\kappa_\nu)$$

where $\bar{\ell} = \ell(-1) = \ell - \frac{\kappa}{|\kappa|}$

These D's are given by

$$\overline{D(\kappa\kappa_\nu)} = j_{\ell_\nu}(qr)\ G_\kappa(pr) - \frac{\kappa_\nu}{|\kappa_\nu|}\ j_{\bar{\ell}_\nu}(qr)\ F_\kappa(qr)$$

$$D(\kappa\kappa_\nu) = j_{\ell_\nu} G_\kappa + \frac{\kappa_\nu}{|\kappa_\nu|} j_{\bar{\ell}_\nu} F_\kappa \quad \text{and}$$

$$D_{JL}(\kappa\kappa_\nu) = (J1L:000)\ \overline{D(\kappa\kappa_\nu)} + \frac{\kappa}{|\kappa|} w_J\ (jj_\nu)(J1L:1\text{-}10)\ D(\kappa\kappa_\nu)$$

where $w_0\ (jj') = 0$, else,

$$w_J\ (jj') = \frac{2j+1+(2j'+1)(-)^{j+j'+J}}{(2J(2J+1))^{\frac{1}{2}}}$$

The j_{ℓ_ν} is the spherical bessel function which arises when the Dirac radial equation is solved for the neutrino, while the F and G arise from the same equation for a spherically symmetric potential, i.e.

$$\frac{dF_\kappa}{dr} = \frac{\kappa-1}{r} F_\kappa - (W-1-V(r))\ G_\kappa(r)$$

$$\frac{dG_\kappa}{dr} = -\frac{\kappa+1}{\nu} G_\kappa + (W+1-V)\ F_\kappa(r)$$

Notice that $A_\mu(JL\kappa\kappa_\nu)$ has the useful property that

$$A_\mu(JL\kappa\text{-}\kappa_\nu) = \mp i \frac{\kappa_\nu}{|\kappa_\nu|} A_\mu(JL\kappa\kappa_\nu) \quad \text{for } e^{\mp}$$

Let us now write more explicitly the expression for the beta-gamma angular correlation function.

$$W(\theta,S) = \frac{1}{2I_i+1} \iint_{\gamma\beta} \sum_{m_f m_i} \Big|\sum_m \int \psi^+(I_f m_f) H_\gamma\ \psi(Im)\ d\tau \int \psi^+(Im) H_\beta\ \psi(I_i m_i)\ d\tau\Big|^2$$

where we have averaged over the initial states (the initial nuclei are unoriented) and summed over the final states. Here $\int_\gamma \int_\beta$ indicate sums over the unmeasured properties such as the electron and neutrino spins and the direction of emission of the neutrino.

W(θ,s) can be factored into two parts, one due to the beta decay and the other due to the electromagnetic transition. We define the following density matrices.

$$<m|\rho_\beta|m'> = \int_\beta \sum_{m_i} \int \psi^+(Im)H_\beta \; \psi(I_i m_i) \; d\tau \left[\int \psi(Im') \; H_\beta \; \psi(I_i m_i) \; d\tau\right]^*$$

$$<m'|\rho_\gamma|m> = \int_\gamma \sum_{m_f} \int \psi^+(I_f m_f)H_\gamma \; \psi(Im) \; d\tau \left[\int \psi^+(I_f m_f) \; H_\gamma \; \psi(Im') \; d\tau\right]^*$$

Using these

$$W(\theta,s) = \frac{1}{2I_i+1} \sum_{mm'} <m|\rho_\beta|m'><m'|\rho_\gamma|m>$$

Similarly, the expression appearing in the shape factor

$$\sum |H_{Ii}|^2 = \frac{1}{2I_i+1} \sum_{mm_i} \int d\Omega_p \int_\beta \int \psi^+(Im)H \; \psi(I_i m_i) \, d\tau \left[\int \psi(Im)H \; \psi(I_i m_i) \; d\tau\right]^*$$

can be rewritten as

$$\sum |H_{Ii}|^2 = \frac{1}{2I_i+1} \sum_{mm'} \delta_{mm'} <m|\rho_\beta|m'>$$

The observables to be calculated in this paper, as well as all other beta decay observables (see Wiedenmuller[11]), can be written in terms of the beta decay density matrix. This matrix is given in Appendix III.

$$<m|\rho_\beta|m'> = \frac{W\,q^2}{p} \sum_{kJJ} (-)^{2I-I_i+m\;J+J'} \frac{(2k+1)}{(2I_i+1)} D^k_{ao}\,(Z\to p)$$

$$\begin{pmatrix} I & I & k \\ m' & -m & a \end{pmatrix} \begin{Bmatrix} I & I & k \\ J & J' & I_i \end{Bmatrix} d_k(JJ')$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$ is a 3j symbol

$\begin{Bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{Bmatrix}$ is a 6j symbol

and $d_k(JJ')$ is defined as

$$d_k(JJ') = (2\pi g)^2 \sum_{\kappa\kappa'\kappa_\nu LL'} \delta_{\ell+\ell'+k,\text{even}}\; e^{i\,\Delta_{\kappa\kappa'}} \rho_J(jj_\nu)\;\rho_{J'}(j'j_\nu)$$

$$\Big((2j+1)(2j'+1)(2J+1)(2J'+1)\Big)^{\frac{1}{2}} (-)^{j'+j+j_\nu-\frac{1}{2}}$$

$$\begin{pmatrix} j & j' & k \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \begin{Bmatrix} J & J' & k \\ J' & j & j \end{Bmatrix} M_{JL}(\kappa\kappa_\nu)\; M^*_{J'L'}(\kappa'\kappa_\nu)$$

The new symbols in this expression are defined as

$$\Delta_{\kappa\kappa'} = \Delta_{\kappa} - \Delta_{\kappa'} \qquad \text{and}$$

$$M_{JL}(\kappa\kappa_{\nu}) = \int r^{2}dr \left\{ D_{1}(JJ\kappa\kappa_{\nu}) \langle Y_{J}\rangle + D_{4}(JJ\kappa\kappa_{\nu}) \langle Y_{J}\vec{\sigma}\cdot\vec{p}\rangle \right.$$

$$\left. + (-)^{L+J+1} \left[D_{2}(JL\kappa\kappa_{\nu}) \langle \vec{V}_{J}{}^{L}\cdot\vec{\sigma}\rangle + D_{3}(JL\kappa\kappa_{\nu}) \langle \vec{V}_{J}{}^{L}\cdot\vec{p}\rangle \right] \right\}$$

where $\langle O_J\rangle$ are called the radial nuclear matrix elements and are independent of m, M, and m_i.

They are defined in the following way.

$$\int \psi^{+}(Im) \left[i^{J} Y_{JM} \right]^{*} \psi(I_{i}m_{i}) \, d\Omega \equiv$$

$$(Im:(i^{J}Y_{JM})^{*}:I_{i}m_{i}) = V.A. \langle Y_{J}\rangle$$

$$(Im:(i^{L}V_{JM}^{L})^{*}\cdot\sigma:I_{i}m_{i}) = V.A. \langle V_{J}{}^{L}\cdot\sigma\rangle$$

$$(Im:(i^{L}V_{JM}^{L})^{*}\cdot p:I_{i}m_{i}) = V.A. \langle V_{J}{}^{L}\cdot p\rangle$$

$$(Im:(i^{J}Y_{JM})^{*}\sigma\cdot p:I_{i}m_{i}) = V.A. \langle Y_{J}\sigma\cdot p\rangle$$

where $V.A. = \sqrt{2I_{i}+1} \; (-)^{J-I-m_{i}} \begin{pmatrix} I & J & I_{i} \\ m & M & -m_{i} \end{pmatrix}$

The relation between these nuclear matrix elements and some of the ones commonly found in the literature are given in Table 1.

Using the above expression for the beta decay density matrix, the

Table 1. Relations Between Symbols‡ Used for Nuclear Matrix Elements

Konopinski and Uhlenbeck	Morita and Morita	Rose and Osborn	This Paper
$\int \vec{r}$	$= M(\vec{r})$	$= \left(\frac{4\pi}{3}\right)^{\frac{1}{2}} \int (I \Vert Y_1 \Vert I_i) r^3 dr$	$= \frac{i\ (-)^{I_i - I}}{3} \left(\frac{4\pi(2I_i+1)}{2I+1}\right)^{\frac{1}{2}} \int \langle Y_1 \rangle\ r^3 dr$
$\int i\vec{\alpha}$	$= -\ i\ M(\vec{\alpha})$	$= \frac{i\ 4\pi}{M\sqrt{3}} \int (I \Vert T_{10}(\hat{r}\vec{p}) \Vert I_i) r^3 dr$	$= \frac{i\ (-)^{I_i - I}}{M} \left(\frac{4\pi(2I_i+1)}{2I+1}\right)^{\frac{1}{2}} \int \langle \vec{V}_1^{\,0} \cdot \vec{p} \rangle\ r^2 dr$
$\int i\ \vec{\sigma} \times \vec{r}$	$= i\ M(\vec{\sigma} \times \vec{r})$	$= \frac{\sqrt{32}\ \pi}{3} \int (I \Vert T_{11}(\hat{r}\vec{\sigma}) \Vert I_i) r^3 dr$	$= i \left(\frac{2}{3}\right)^{\frac{1}{2}} (-)^{I_i - I} \left(\frac{2I_i+1}{2I+1}\right)^{\frac{1}{2}} \int \langle \vec{V}_1^{\,1} \cdot \vec{\sigma} \rangle\ r^3 dr$
$\int B_{ij}$	$= M(B_{ij})$	$= \frac{8\pi}{3} \int (I \Vert T_{21}(\hat{r}\vec{\sigma}) \Vert I_i) r^3 dr$	$= \frac{i\ 2}{\sqrt{3}} (-)^{I_i - I} \left(\frac{2I_i+1}{2I+1}\right)^{\frac{1}{2}} \int \langle \vec{V}_2^{\,1} \cdot \vec{\sigma} \rangle\ r^3 dr$

‡These are, respectively, the symbols used by Konopinski and Uhlenbeck,[12] Morita and Morita,[4] Rose and Osborn,[10] and this paper.

shape can be rewritten as

$$C = \frac{2\pi}{\hbar} \sum |H_{Ii}{}^{\beta}|^2 = \frac{2\pi}{\hbar} \frac{1}{2I_i+1} \sum_{mm'} \int d\Omega p \ \delta_{mm'} \langle m|\rho_\beta|m'\rangle$$

$$= \frac{2^3\pi^2}{\hbar} \frac{W\ q^2}{p} \sum_J \frac{(-)^J}{(2J+1)^{\frac{1}{2}}} \ d_0(JJ) = \frac{2^5\pi^4 g^2}{\hbar} \sum_{\kappa\kappa_\nu J} \Big| \sum_L \rho_J(jj_\nu)\ M_{JL}(\kappa\kappa_\nu)\Big|^2$$

Using Frauenfelder's expression for the gamma decay density matrix element, as shown in Appendix III, the beta-gamma angular correlation function can be written as

$$W(\theta,s) = \frac{1}{2I_i+1} \sum_{mm'} \langle m|\rho_\beta|m'\rangle \langle m'|\rho_\gamma|m\rangle = \sum_k S^k\ A_k{}'\ P_k\ (\cos\ \theta)$$

$$= \frac{(-)^{I-I_i-2I_f}}{(2I+1)^{\frac{1}{2}}} \sum_k (-)^k S^k\ (2k+1)^{\frac{1}{2}} \sum_{JJ'} \ (-)^{J+J'} \begin{Bmatrix} II & k \\ JJ' & I_i \end{Bmatrix} d_k(JJ')$$

$$\Big[\sum_{L_\gamma L_\gamma{}'} (-)^{L_\gamma - L_\gamma{}'} F_k(L_\gamma L_\gamma{}' I_f I)\ \langle I_p \| L_\gamma \| I\rangle \langle I_p \| L_\gamma{}' \| I\rangle^* \Big] P_k(\cos\theta)$$

Using this we find that $A_0{}'$ is proportional to the shape.

$$A_0{}' = \frac{1}{2^3\pi^2} \frac{C(E)}{2I+1} \sum_{L_\gamma} \ | \langle I_p \| L_\gamma \| I\rangle \ |^2$$

and hence

$$A_k = \frac{W\ q^2}{p} \frac{A_k\ (\gamma) \quad 2^3\pi^2}{C(E)} (2I+1)^{\frac{1}{2}}\ (-)^{I-I_i-2I_f}\ (-)^k\ (2k+1)^{\frac{1}{2}} \times$$

$$\times \sum_{JJ'} (-)^{J+J'} \begin{Bmatrix} I & I & k \\ J & J' & I_i \end{Bmatrix} d_k(JJ')$$

where $A_k(\gamma)$ is defined as

$$A_k(\gamma) = \frac{\sum\limits_{L_\gamma L_\gamma'} (-)^{L_\gamma - L_\gamma'} F_k(L_\gamma L_\gamma' I_f I) \langle I_f \| L_\gamma \| I \rangle \langle I_f \| L_\gamma' \| I \rangle^*}{\sum\limits_{L_\gamma} | \langle I_f \| L_\gamma \| I \rangle |^2}$$

Now let us apply this formalism to $1^-(\beta)2^+(\gamma)0^+$ decay (see Figure 1). Since, $|I-I_i| \leq J \leq I+I_i$, J takes on only the values 1, 2, 3. Also, since $|I-I_f| \leq L_\gamma \leq I+I_f$, L_γ has only the value 2. Since there is no change in parity between the intermediate and final states, we have pure E2 electromagnetic radiation. There is a change in parity between the initial and intermediate state, thus only the odd parity nuclear matrix elements contribute. The parity of the various matrix elements is given in Table 2.

Now, we make the same approximations as Morita and Morita.[4]

$$\sum_L M_{1L} (\kappa\kappa_\nu) \simeq \int r^2 dr \Big\{ i\, C_V A_4 \, (11\kappa\kappa_\nu) \langle Y_1 \rangle$$

$$-i\, C_A\, A(11\kappa\kappa_\nu) \langle V_1{}^1 \cdot \sigma \rangle + \frac{C_V}{Mc} A(10\kappa\kappa_\nu) \langle V_1{}^0 \cdot p \rangle \Big\}$$

$$\sum_L M_{2L} (\kappa\kappa_\nu) \simeq \int r^2 dr \; i\, C_A\, A(21\kappa\kappa_\nu) \langle V_2{}^1 \cdot \sigma \rangle$$

$$\sum_L M_{3L} (\kappa\kappa_\nu) \simeq 0$$

$$_{Z+1}X^{A} \rightarrow {}_{Z}Y^{A^*} + e^- + \bar{\nu}$$

$$_{Z}Y^{A^*} \rightarrow {}_{Z}Y^{A} + \gamma$$

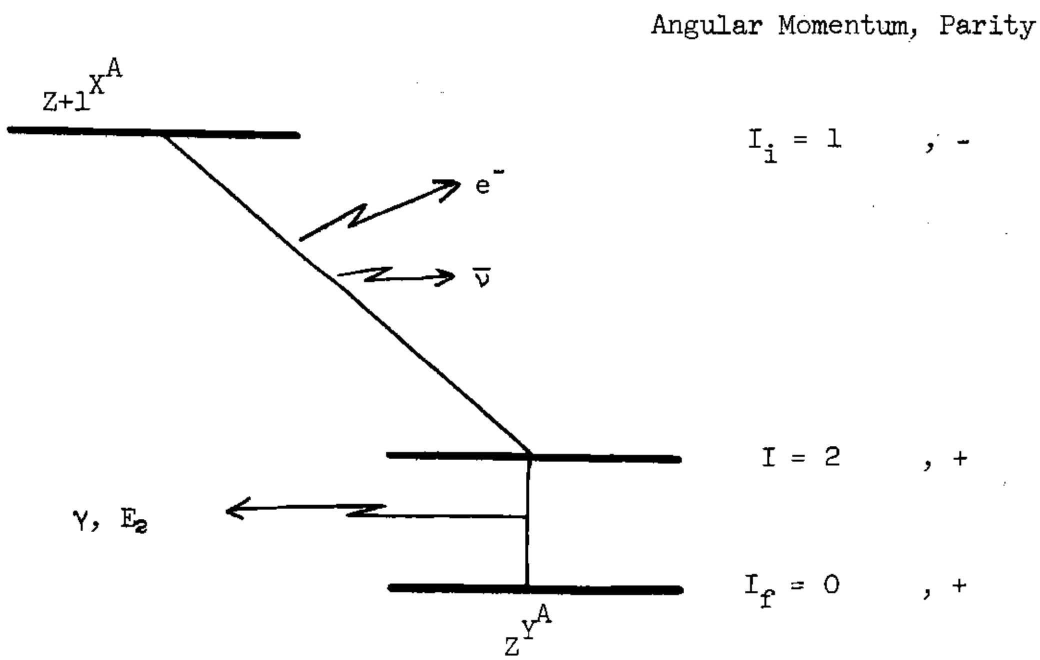


Figure 1. The Decay Scheme $1^-(\beta)\ 2^+(\gamma)\ 0^+$

Table 2. Parity of the Nuclear Operators

$< Y_J >$	$(-)^J$
$< V_J^L \cdot \sigma >$	$(-)^L$
$< V_J^L \cdot p >$	$(-)^{L+1}$
$< Y_J \ \sigma \cdot p >$	$(-)^{J+1}$

$$D_1(JJ\kappa\kappa_\nu) \simeq \delta_{L,J}\, i^L\, C_V\, A_4(1\kappa\kappa_\nu)$$

$$D_2(JL\kappa\kappa_\nu) \simeq i^L\, C_A\, A(JL\kappa\kappa_\nu)$$

that the terms j_1G_2, j_1F_{-2}, j_2G_1, j_2F_{-1}, j_1G_1, j_1F_{-1}, j_1F_2, and j_1G_{-2}, which appear in the $A(JL\kappa\kappa_\nu)$'s, can be neglected

that $j_\ell\,(Qr) \simeq (Qr)^\ell/(2 + 1)!!$

that
$$\sum_{\substack{\kappa\ = \pm 1\ \pm 2\ \pm 3\ \ldots \\ \kappa'\ \quad '' \\ \kappa_\nu\ \quad ''}} \simeq \sum_{\substack{\kappa\ = \pm 1,\ \pm 2 \\ \kappa'\ \quad '' \\ \kappa_\nu\ \quad ''}}$$

and finally what will be called the radial approximation

$$\int A_i\ (JL\kappa\kappa_\nu) <\sigma_J{}^L> r^2 dr \simeq \left[\frac{A_i(JL\kappa\kappa_\nu)}{r^L}\right]_{Rp} \int <\sigma_J{}^L> r^{2+L}\, dr$$

we get the same results as Morita and Morita as shown in Appendix IV.

$$C(E) = \frac{d\lambda}{dW} = \frac{5}{3}\ (2\pi g)^2\ C_{MM}\ F(Z,E)\ E\ q^2\ p$$

where C_{MM} is their expression for the correction factor for the beta spectrum. Hence the ratios are the same, i.e.

$$\frac{C(E)}{C(E_R)} = \frac{C_{MM}(E)}{C_{MM}(E_R)}$$

Similarly
$$A_2 = \frac{A_2{}'(MM)}{A_0{}'(MM)}$$

CHAPTER II

THE RADIAL APPROXIMATION

Separation of the Nuclear from the Lepton Contribution

The radial approximation

$$\int_0^\infty A_i(JLKK_\nu)\, \langle O_J^{\,L}\rangle\, r^2 dr \simeq \left[\frac{A_i(JLKK_\nu)}{r^L} \right]_{R_p} \int_0^\infty \langle O_J^{\,L}\rangle\, r^{2+L}\, dr$$

is made to separate the lepton contribution A_i, i.e., $A_4(JJKK_\nu)$ or $A(JLKK_\nu)$, from the nuclear contribution $\langle O_J^{\,L}\rangle$, i.e., $\langle Y_J\rangle$, $\langle V_J^{\,L}\cdot\sigma\rangle$, $\langle V_J^{\,L}\cdot p\rangle$, or $\langle Y_J \sigma\cdot p\rangle$, in the beta decay matrix elements. The reason for this is that the nuclear contribution depends on the nuclear model and is not accurately known. Hence the desire is to separate out the nuclear contributions in the expressions for the beta decay observables and use the experimental data to determine these nuclear terms and in turn use these to evaluate nuclear models.

In order to facilitate the discussion of the radial integral, let us use the radius of the nucleus as the unit length, i.e.,

$$r_{NS} \simeq 1.2\ A^{\frac{1}{3}} \text{ in fermis} \qquad\qquad \text{or}$$

$$r_{NS} \simeq \frac{.4285}{137.03}\ A^{\frac{1}{3}}$$

in rational relativistic electron units, where A is the number of nucleons in the nucleus.

Realizing that the nuclear wave functions will go to zero rapidly outside the above defined nuclear surface, one may replace the limits on the radial integral as in the equation below.

$$\int_0^\infty A_i(JLK\kappa_\nu) \langle O_J^L \rangle r^2 dr = \int_0^{r_E} A_i(JLK\kappa_\nu) \langle O_J^L \rangle r^2 dr$$

where r_E is greater than r_{NS} but probably smaller than $2r_{NS}$. To further simplify this integral using the properties of the nuclear contribution, one needs to base his arguments on an explicit nuclear model.

Next, let us direct our attention to the lepton contribution to the radial integral. As seen in Appendix II, the various $A_i(JLK\kappa_\nu)$'s are combinations of the spherical bessel functions and the F's and G's are the solutions to the Dirac equation for the electron in a spherical potential well.

The expressions for the spherical bessel functions and their small argument limits are given in Table 3. From them, it is easy to show for a certain explicit case that the approximation

$$j_\ell (qr) \simeq (qr)^\ell/(2\ell + 1)!!$$

is a good approximation for a typical beta decay.

Consider the case when a nucleus of A = 186 nucleons emits an electron of maximum kinetic energy, 934 keV. The energy of the electron is given by

$$E = 1 + \frac{KE}{511 \text{ keV}} = (1 + p^2)^{\frac{1}{2}}$$

$$p_{max} = 2.65$$

Table. 3. Small Argument Limit of the Spherical Bessel Functions

$$j_0(qr) = \frac{\sin qr}{qr} \xrightarrow[qr \to 0]{} 1 - \frac{(qr)^2}{6} + \cdots$$

$$j_1(qr) = \frac{\sin qr}{(qr)^2} - \frac{\cos qr}{qr} \longrightarrow \frac{qr}{3}\left[1 - \frac{(qr)^2}{10} + \cdots\right]$$

$$j_2(qr) = \left[\frac{3}{(qr)^2} - 1\right]\frac{\sin qr}{qr} - \frac{3\cos(qr)}{(qr)^2} \longrightarrow \frac{(qr)^2}{15}\left[1 - \frac{(qr)^2}{14} + \cdots\right]$$

$$j_\ell(qr) \longrightarrow \frac{(qr)^\ell}{(2\ell + 1)!!}$$

The neutrino's momentum is given by the expression for the total energy after collision

$$E_T = 1 + \frac{KE_{max}}{511\ keV} + T_R = E + E_\nu + T_R = (1 + p^2)^{\frac{1}{2}} + q + T_R$$

The recoil energy, T_R, of the nucleus can be neglected, since the maximum value it can have is when the neutrino carries off zero energy.

$$T_{RM} = \frac{1}{2} \frac{p_M^2}{1837} \simeq 10^{-5}$$

If the electron carries off a relatively small momentum $p = 0.6$, the neutrino will carry off $q = 1.66$, and we see that the largest term in the spherical bessel functions which we would be neglecting, $(qr)^2/6$, would contribute less than 0.1 percent error even at r equal 2.6 nuclear surfaces.

The electron contributions to the radial integral are given by the F's and G's which are solutions to the Dirac radial equations

$$\frac{dF_\kappa}{dr} = \frac{\kappa - 1}{r} F_\kappa - (W - 1 - V)\ G_\kappa$$

$$\frac{dG_\kappa}{dr} = - \frac{\kappa + 1}{r} G_\kappa + (W - 1 - V)\ F_\kappa$$

Uniform Charge Distribution

If it is assumed that the emitted electron sees a uniformly charged unscreened daughter nucleus of charge Z , then the potential is

$$V = \begin{cases} -\dfrac{\alpha Z}{2r_{NS}}\left(3 - \left[\dfrac{r}{r_{NS}}\right]^2\right) & r \leqq r_{NS} \\ -\dfrac{\alpha Z}{r} & r \geqq r_{NS} \end{cases}$$

The analytic solution of this pair of coupled first order differential equations outside the nuclear surface is given by Rose[13] as a sum of the regular and irregular coulomb solutions, e.g.,

$$F_\kappa^{out} = A\, R_\kappa^{out}\,(r) + B\, I_\kappa^{out}\,(r)$$

$$G_\kappa = \frac{\dfrac{dF_\kappa}{dr} - \dfrac{\kappa - 1}{r} F_\kappa}{-\,(W - 1 + \alpha Z/r)}$$

Inside the nuclear surface there is only a regular solution, since rF and rG have to be finite at r = 0.

$$F^{in}(r) = N\, R^{in}(r)$$

The constants of integration A and B can be found in terms of N by equating the functions and their derivatives at the nuclear surface, i.e.,

$$N\, R^{in}(r_{NS}) = A\, R^{out}(r_{NS}) + B\, I^{out}(r_{NS})$$

$$N\left(\frac{dR^{in}}{dr}\right)_{r_{NS}} = A\left(\frac{dR^{out}}{dr}\right)_{r_{NS}} + B\left(\frac{dI^{out}}{dr}\right)_{r_{NS}}$$

and N can be found by normalization. Bhalla and Rose[14] have, for this potential, tabulated the value for F and G at the nuclear surface for $\kappa = \pm 1, \pm 2$ and for extensive values of A, Z, and p.

In terms of the inside and outside solutions, the radial integral can be rewritten as

$$\int_0^\infty A_i(JL\kappa\kappa_\nu) \langle O_J^L \rangle r^2 dr = \int_0^{r_{NS}} A_i^{in}(JL\kappa\kappa_\nu) \langle O_J^L \rangle r^2 dr$$

$$+ \int_{r_{NS}}^\infty A_i^{out}(JL\kappa\kappa_\nu) \langle O_J^L \rangle r^2 dr$$

expecting the second term to be small since $\langle O_J^L \rangle$ should go to zero fairly rapidly outside the nuclear surface.

Now let us look at the inside solutions in more detail dropping the superscript "in".

From Rose for $\kappa > 0$

$$G_\kappa = \frac{a_0}{r_{NS}} \left(\frac{r}{r_{NS}}\right)^{|\kappa|} \left[1 + \frac{a_1}{a_0}\left(\frac{r}{r_{NS}}\right)^2 + \frac{a_2}{a_0}\left(\frac{r}{r_{NS}}\right)^4 + \dots\right]$$

$$F_\kappa = \frac{b_0}{r_{NS}} \left(\frac{r}{r_{NS}}\right)^{|\kappa|-1} \left[1 + \frac{b_1}{b_0}\left(\frac{r}{r_{NS}}\right)^2 + \frac{b_2}{b_0}\left(\frac{r}{r_{NS}}\right)^4 + \dots\right]$$

where

$$a_0(\kappa) = \frac{r_{NS}(W+1) + 3\alpha Z/2}{2|\kappa| + 1} b_0(\kappa)$$

$$a_n = b_n = 0 \quad \text{for} \quad n < 0$$

$$b_{n+1}(\kappa) = \frac{\frac{\alpha Z}{2} a_{n-1}(\kappa) - [r_{NS} (W - 1) + 3\alpha Z/2] a_n(\kappa)}{2(n + 1)}$$

$$a_n(\kappa) = \frac{[r_{NS} (W + 1) + \frac{3\alpha Z}{2}] b_n(\kappa) - \frac{\alpha Z}{2} b_{n-1}(\kappa)}{2|\kappa| + 2n + 1}$$

For $\kappa < 0$

$$G_\kappa \rightarrow F_\kappa$$

$$F_\kappa \rightarrow G_\kappa$$

$$W \rightarrow -W$$

$$Z \rightarrow -Z$$

For $\kappa > 0$, b_0 equals the normalization constant N. Hence, for the inside solution, we have a power series in $\left(\frac{r}{r_{NS}}\right)^2$ with decreasing coefficients $\frac{a_n}{a_0}$ and $\frac{b_n}{b_0}$.

Using these expansions for the inside electron wave functions and keeping only the largest term, we see that the A's appearing in Appendix II, Table 15 $A_i^{in} (JL\kappa\kappa_\nu) \sim r^L$, for a one term expansion.

Looking at the explicit case tabulated in Table 4, F_{-1} is only linear in r to 25 percent at the nuclear surface. Hence some of the terms

$$\frac{A_i^{in} (JL\kappa\kappa_\nu)}{r^L}$$

will be constant in r to only 25 percent near the nuclear surface.

If the approximation that

Table 4. The Expansion Coefficients for a Particular Case

$A = 186$, $r_{NS} = 0.0179$, $K = -1$	$Z = 76$, $p = .6$	$Q = 0.35$, $W = (p^2 + 1)^{\frac{1}{2}}$, $p = 2.6$
$\frac{a_0}{b_0}$	-0.278	-0.288
$\frac{b_1}{b_0}$	-0.121	-0.129
$\frac{a_1}{a_0}$	-0.272	-0.270
$\frac{b_2}{b_0}$	0.0358	0.0375
$\frac{a_2}{a_0}$	0.0356	0.0339
$\frac{b_3}{b_0}$	-0.00494	-0.00505
$\frac{a_3}{a_0}$	-0.00560	-0.00569
$\frac{b_4}{b_0}$	0.000513	0.000523
$\frac{a_4}{a_0}$	0.000587	0.000587

$$\sum_{0}^{4} \frac{b_n(-1)}{b_0(-1)} = 0.910 \qquad \sum_{0}^{4} \frac{a_n(-1)}{a_0(-1)} = 0.759$$

$$\frac{A_i^{in}(JLK\kappa_\nu)}{r^L} = \text{constant}$$

is used, it does not mean that the beta decay observables calculated using this approximation necessarily will be as inaccurate as the approximation itself. First, the radial approximation

$$\int_0^\infty \frac{A_i(JLK\kappa_\nu)}{r^L} <O_J^L> r^{2+L}\, dr \simeq \left[\frac{A_i(JLK\kappa_\nu)}{r^L}\right]_{r=r_p} \int_0^\infty <O_J^L> r^{2+L}\, dr$$

also depends on the explicit form of the nuclear contribution. Second, the beta decay observables depend on sums of products of these radial integrals and hence the approximation errors might cancel out in summation. Third, we might expect that the normalized shape factor and beta-gamma A_2 coefficient might not be too sensitive to such an approximation because they depend on ratios of sums of products of these radial integrals and hence the approximation errors may divide out. Therefore, we might expect the half life calculation, since it does not involve ratios to be most sensitive to approximations.

Another approximation that is used to evaluate this radial integral (this will be called the Buhring approximation[15]), is the following

$$\int_0^\infty A_i(JLK\kappa_\nu) <O_J^L> r^2 dr = \int_0^{r_{NS}} A_i^{in}(JLK\kappa_\nu) <O_J^L> r^2 dr + \int_{r_{NS}}^\infty A_i^{out}(JLK\kappa_\nu) <O_J^L> r^2 dr$$

$$= \int_0^\infty A_i^{in}(JL\kappa\kappa_\nu) < O_J^L > r^2 dr + \int_{r_{ns}}^\infty \left[A_i^{out}(JL\kappa\kappa_\nu) - A_i^{in}(JL\kappa\kappa_\nu) \right] < O_J^L > r^2 dr$$

$$\approx \int_0^\infty A_i^{in}(JL\kappa\kappa_\nu) < O_J^L > r^2 dr$$

Where $A_i^{in}(JL\kappa\kappa_\nu)$ evaluated outside the nuclear surface means that one uses the power series expansion for the electrons F and G wave functions, which are valid inside the nucleus, as if they were also valid outside the nuclear surface.

$$A_i^{out}(JL\kappa\kappa_\nu) \simeq A_i^{in}(JL\kappa\kappa_\nu)$$

From Appendix II, Table 15, it is seen that for those $A_i(JL\kappa\kappa_\nu)$'s

$$A_i(JL\kappa\kappa_\nu) = r^L \sum_{n=o}^{\infty} {}^n\alpha_i(JL\kappa\kappa_\nu) \left(\frac{r}{r_{NS}}\right)^{2n}$$

Using this expansion, Damgard and Winther[16] have developed expressions for the beta decay matrix elements. For example

$$A(11\text{-}11) = -\sqrt{2}\,(j_0 F_{-1} + j_1 G_{-1})$$

$$A^{in}(11\text{-}11) = -\sqrt{2}\,a_0(-1)\frac{r}{r_{NS}} \sum_{n=o}^{\infty} \left(\frac{r}{r_{NS}}\right)^{2n} \left(\frac{1}{r_{NS}}\frac{a_n}{a_0} + \frac{Q}{3}\frac{b_0}{a_0}\frac{b_n}{b_0}\right)$$

For the explicit case tabulated in Table 4 for p = 2.6, this would be

$$A^{in}(11\text{-}11) = 15.7\sqrt{2}\, a_0 \frac{r}{r_{NS}} \left[1 - .27 \left(\frac{r}{r_{NS}}\right)^2 + .0355 \left(\frac{r}{r_{NS}}\right)^4 + \cdots\right]$$

To check the validity of these approximations for the radial integral, we will compare their results with that obtained by evaluating the radial integral exactly using numerical techniques. To do this we first need exact values of the electron wave functions both inside and outside the nuclear surface. These are obtained numerically by using sixth order Runge Kutta Nystrom difference equations techniques.(17)

The simplest difference equation technique which can be used to solve a differential equation is the following. The equation to be solved for f, where A(r) and B(r) are known functions, is

$$\frac{df}{dr} = A(r)\ f(r) + B(r) = \lim_{\Delta r \to 0} \frac{f(r + \Delta r) - f(r)}{\Delta r}$$

Let $\Delta r = H$ and $r_n = nH + r_0$, where $n = 0,1,2,\ \ldots$; hence, if H is small enough

$$\frac{f(r_n) - f(r_{n-1})}{H} = A(r_{n-1})\ f(r_{n-1}) + B(r_{n-1})$$

Since this is a first order differential equation and there is one constant of integration, another condition is necessary to uniquely determine its solution. For instance

$$\left(\frac{df}{dr}\right)_{r=0} = 0 = \frac{f(r_1) - f(r_0)}{H}$$

Now $f(r_0)$ can be calculated, then $f(r_1)$, then $f(r_2)$, and so forth. Hence $f(r)$ can be found numerically and the accuracy of its value at $r_n = nH + r_0$ depends upon the size of the step H.

To determine, for our case, the F and G electron wave functions, we have a pair of first order coupled differential equations and hence have two constants of integration. The explicit equations which were solved using the sixth order Runge-Kutta Nystrom method are the following. For $\kappa = \pm 1$

$$\frac{dU_1}{dr} = -\frac{\kappa}{r} U_1 + (W + 1 - V(r))\, U_2(r,\kappa)$$

$$\frac{dU_2}{dr} = \frac{\kappa}{r} U_2 - (W - 1 - V)\, U_1(r,\kappa)$$

where $$U_1(r,\kappa) \equiv rG_\kappa$$

and $$U_2(r,\kappa) \equiv rF_\kappa$$

For $\kappa = \pm 2$

$$\frac{dG_\kappa}{dr} = -\frac{\kappa + 1}{r} G_\kappa + (W + 1 - V)\, F_\kappa$$

$$\frac{dF_\kappa}{dr} = \frac{\kappa - 1}{r} F_\kappa - (W - 1 - V)\, G_\kappa$$

The boundary conditions used at r equal zero were for $\kappa = 1$ $\qquad \frac{dU_1}{dr} = 0$ and $\frac{dU_2}{dr} \neq 0$;

for $\kappa = -1$ $\qquad \frac{dU_1}{dr} \neq 0$ and $\frac{dU_2}{dr} = 0$;

for $\kappa = 2$ $\qquad \frac{dG_2}{dr} = 0$ and $\frac{dF_2}{dr} \neq 0$;

for $\kappa = -2$ $\qquad \frac{dG_{-2}}{dr} \neq 0$ and $\frac{dF_{-2}}{dr} = 0$

Initially, the non-zero values of the derivatives at $r = 0$ were guessed, and then the digital computer stepped out the values of F and G to the nuclear surface where their values are known from the Bhalla and Rose tables.(14) Then the initial guess was divided by the ratio of the trial solution at the nuclear surface to the Bhalla and Rose solution of the larger in magnitude of either F or G. Then F and G were recalculated giving values which have the same normalization as those of Bhalla and Rose, namely

$$\int_0^1 (F_\kappa^2 + G_\kappa^2)\, r^2 dr = 1$$

The step size H was chosen by trial. When it was halved, the ratio of F to G at the nuclear surface did not change by more than one part in 10^6. This step size turned out to be about 1/64 of a nuclear surface. With such a step size, this procedure gave results such that the larger in magnitude of either F or G was equal to the Bhalla and Rose value. The smaller, then, was within 0.02 percent of their value.

Figure 2 shows typical plots of F and G. The radial approxima-

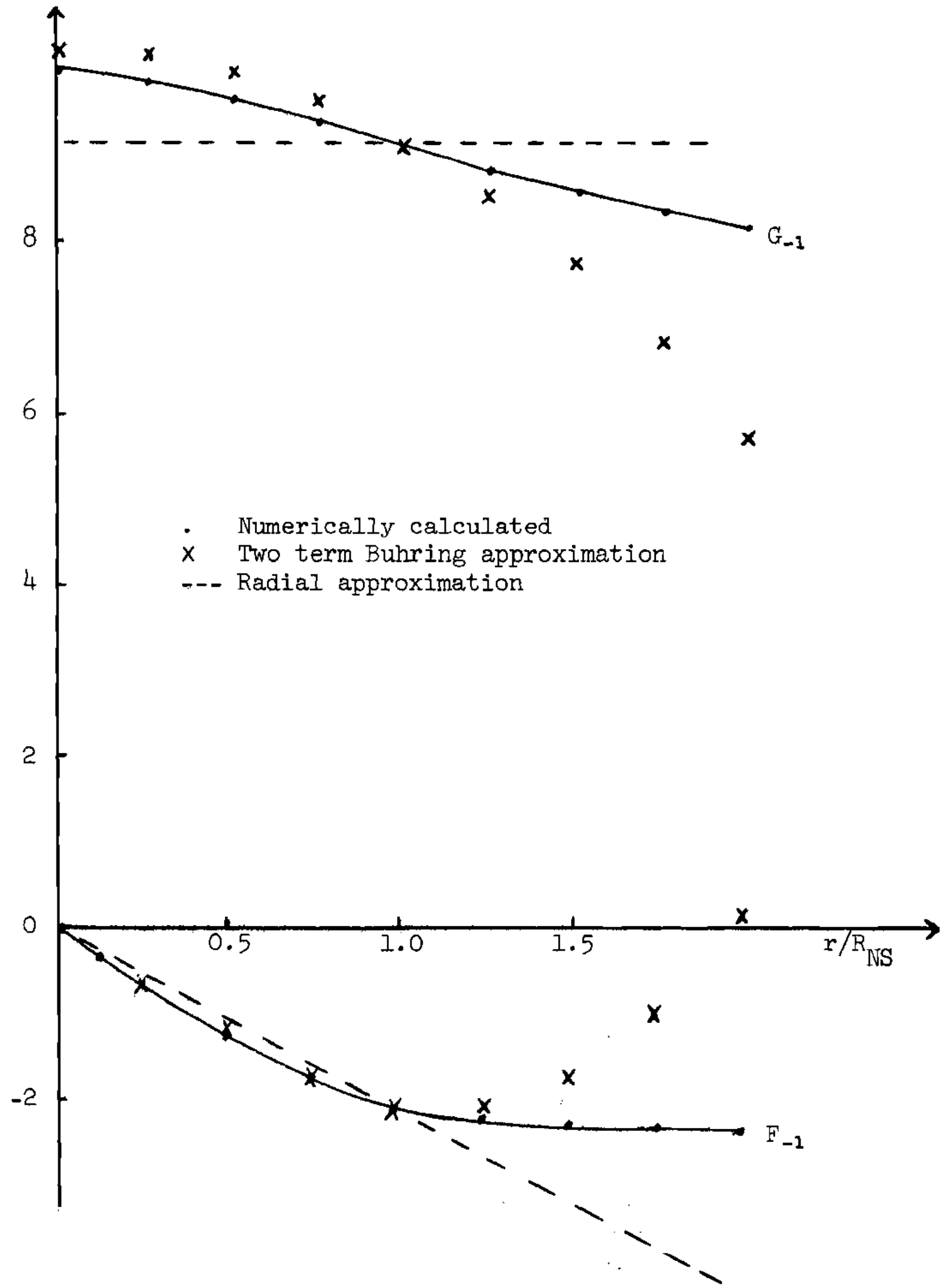


Figure 2. Graphs of the Radial Electron Wave Functions for A = 170, Z = 70, p = 2.4, T_{max} = 967 keV, and for a Uniform Charge Distribution

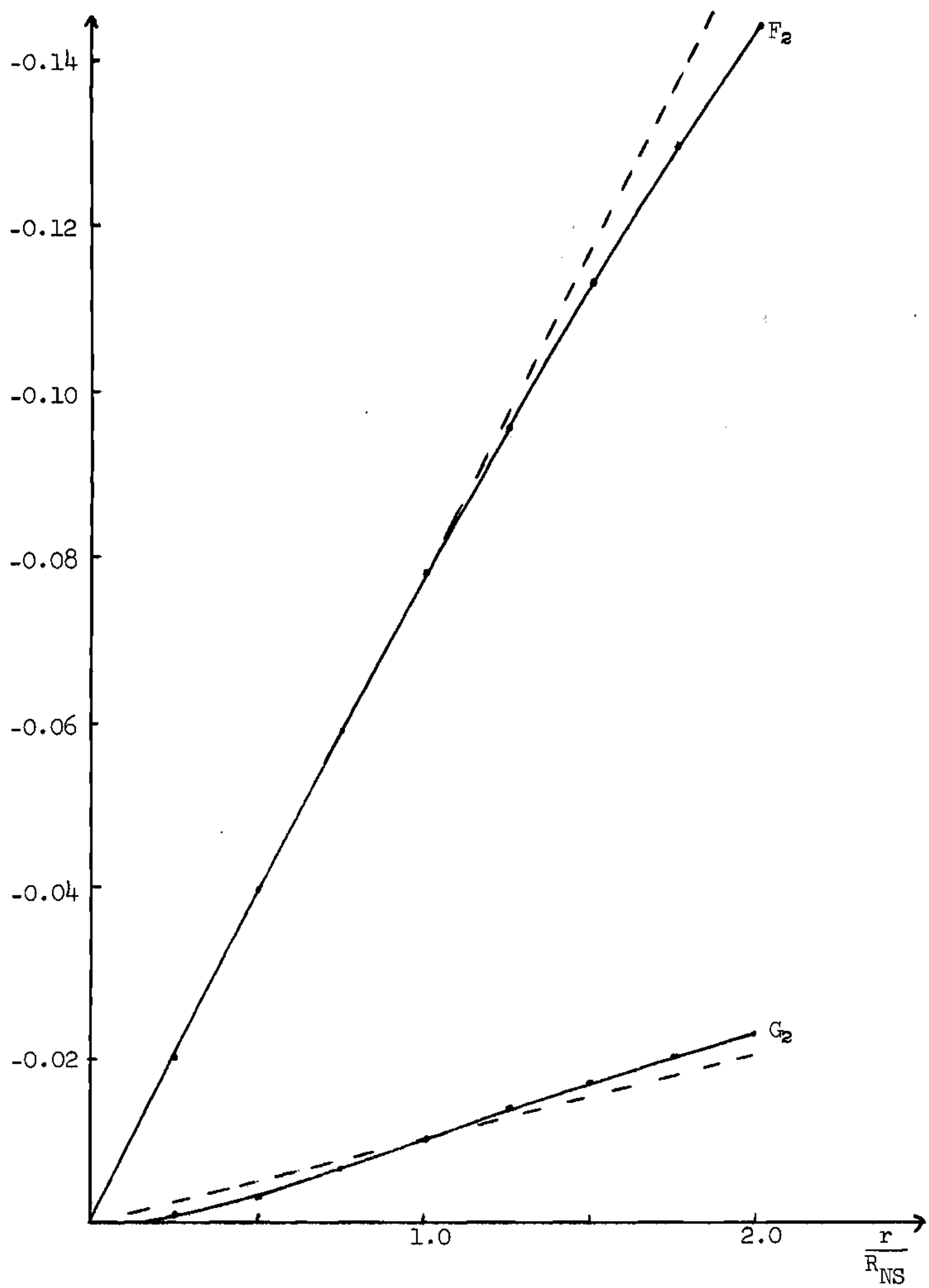


Figure 2. Continued

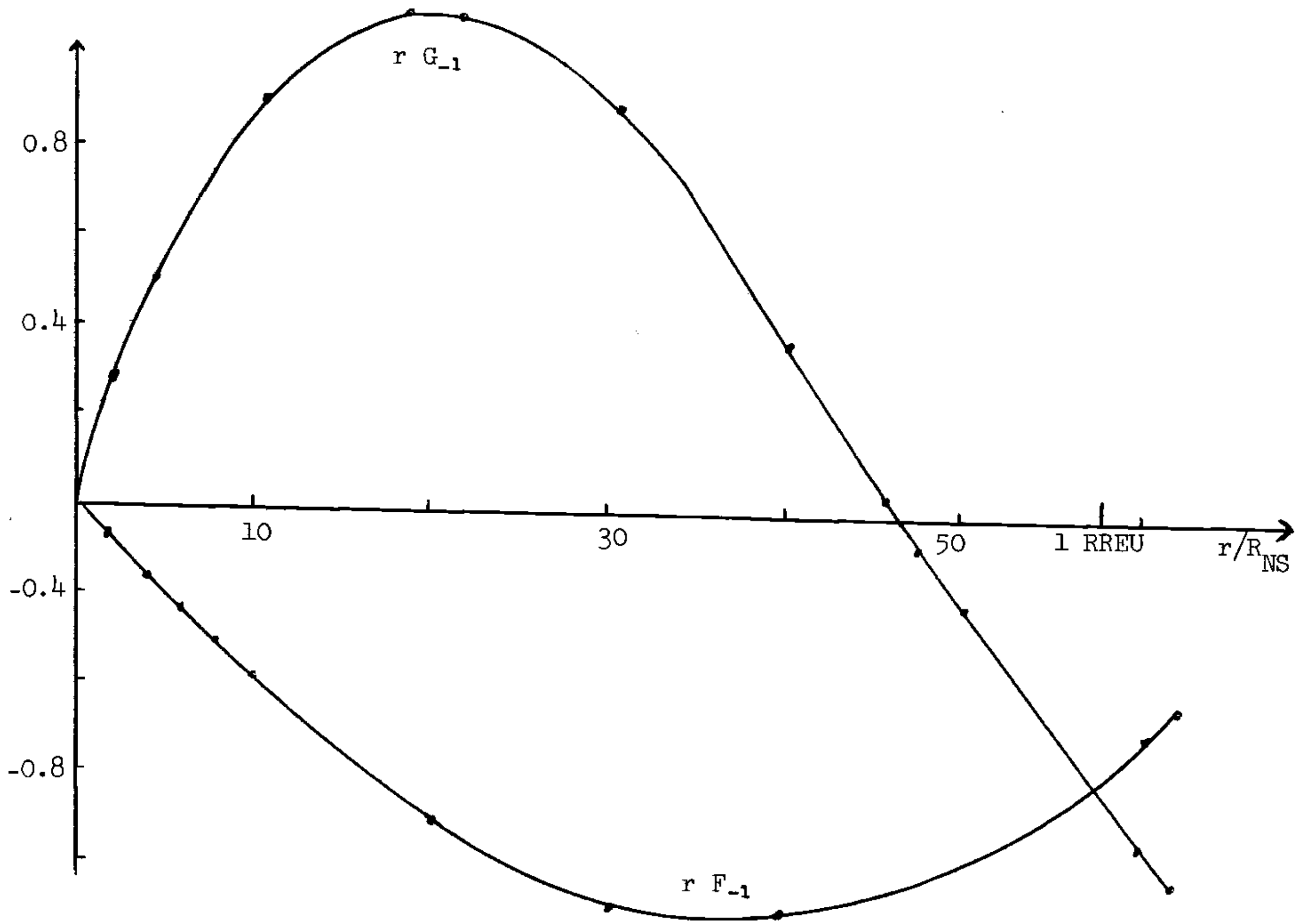


Figure 2. Continued

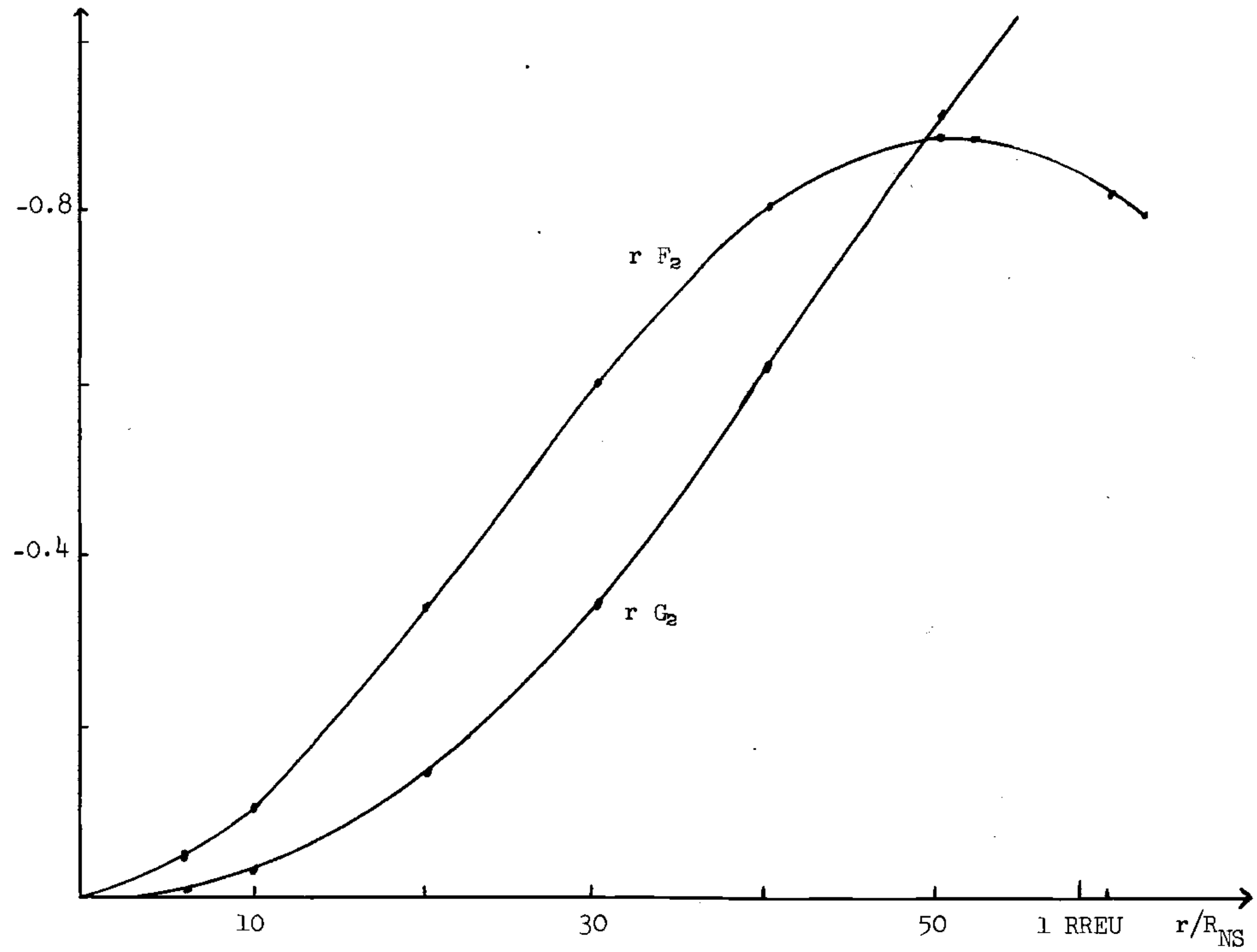


Figure 2. Concluded

tion usually evaluates the lepton contribution at the nuclear surface. The straight dashed lines are the values which the radial approximation uses, the solid curves are obtained by the numerical methods indicated above.

F_1 has a similar shape to G_{-1} $\quad (F_1 \sim -\frac{1}{1.4} G_{-1})$.

G_1 has a similar shape to F_{-1} $\quad (G_1 \sim \frac{1}{1.4} F_{-1})$.

G_{-2} has a similar shape to F_2 $\quad (G_{-2} \sim 1.4\, F_2)$.

F_{-2} has a similar shape to G_2 $\quad (F_{-2} \sim 1.4\, G_2)$.

Since for large r, F and G should go like 1/r times a sinisoidal function, in Figure 2 the values of rF and rG are plotted for the same case out to r = 1 rational relativistic electron units.

Also in Figure 2, the x's are a plot of the first two terms in the Buhring expansion, i.e.,

$$G_{-1} = \frac{b_0}{r_{NS}} \left[1 + \frac{b_1}{b_0} \left(\frac{r}{r_{NS}}\right)^2\right]$$

$$F_{-1} = \frac{b_0}{r_{NS}} \frac{a_0}{b_0} \frac{r}{r_{NS}} \left[1 + \frac{a_1}{a_0} \left(\frac{r}{r_{NS}}\right)^2\right]$$

where $\frac{b_0}{r_{NS}}$ is determined by setting G_{-1} $(r = r_{NS})$ equal to the Bhalla and Rose value.

<u>Fermi Charge Distribution (Hofstadter Potential)</u>

The above numerical calculation for finding the electron wave function, for the electron seeing a uniform nuclear charge, is rather easy to extend to a more realistic spherical charge distribution. Hence in Figure 5 there is a plot of F and G when the electron sees a Fermi charge distribution,[18] i.e.,

$$\rho_F = \frac{N}{4\pi} \frac{1}{1 + e^{(r-c)/a_0}}$$

which experimentally fits the Hofstadter scattering data. A plot of this charge distribution is given in Figure 3.

N can be determined for a spherical charge density from

$$Ze = \int_0^\infty \rho \; 4\pi \; r^2 dr$$

hence

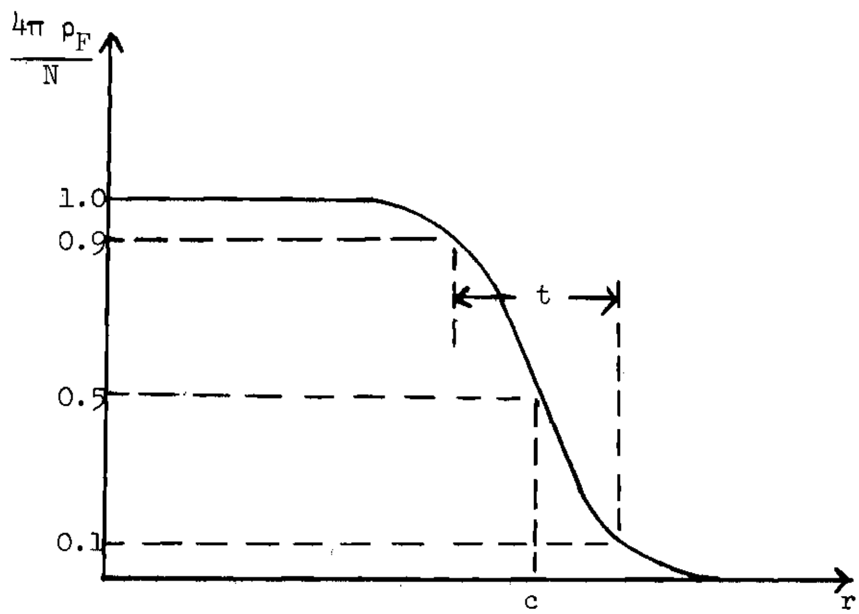
$$N = \frac{Ze}{\int_0^\infty \frac{r^2 dr}{1 + e^{(r-c)/a_0}}}$$

The potential energy for an electron seeing this spherical charge distribution can be found by numerically solving

$$V(r) = -4\pi e \left[\frac{1}{r} \int_0^r \rho_F(x) \; x^2 dx + \int_r^\infty \rho_F(x) \; x dx\right]$$

On Figure 4, we have this potential plotted in units of .511 MeV and compared with the uniform potential as well as the point charge potential

$V = -\frac{Ze^2}{r} = -\frac{\alpha Z}{r}$ in rational relativistic electron units.



$$\rho_F = \frac{4\pi\ N}{1 + e^{(r-c)/a_o}}$$

c = the half density radius

t = the skin thickness

a_o = t/ln 81

For $50 \leqq A \leqq 200$

$c = (1.07 \pm 0.02)\ A^{\frac{1}{3}}$ fermis

$t = (2.4 \pm 0.3)$ fermis

Figure 3. The Fermi Charge Distribution

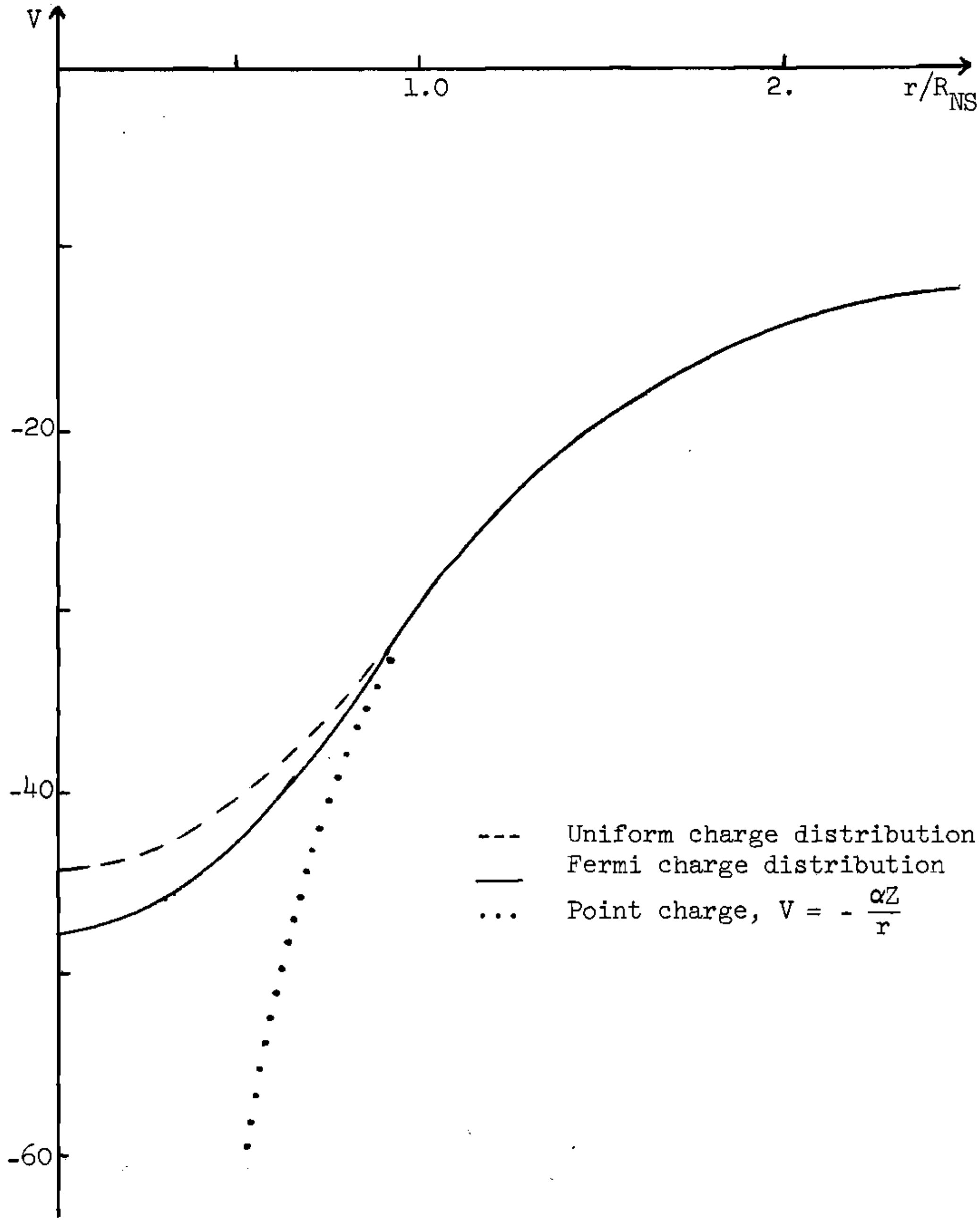


Figure 4. The Potential for a Uniform Charge, Point Charge, and a Fermi Charge Distribution

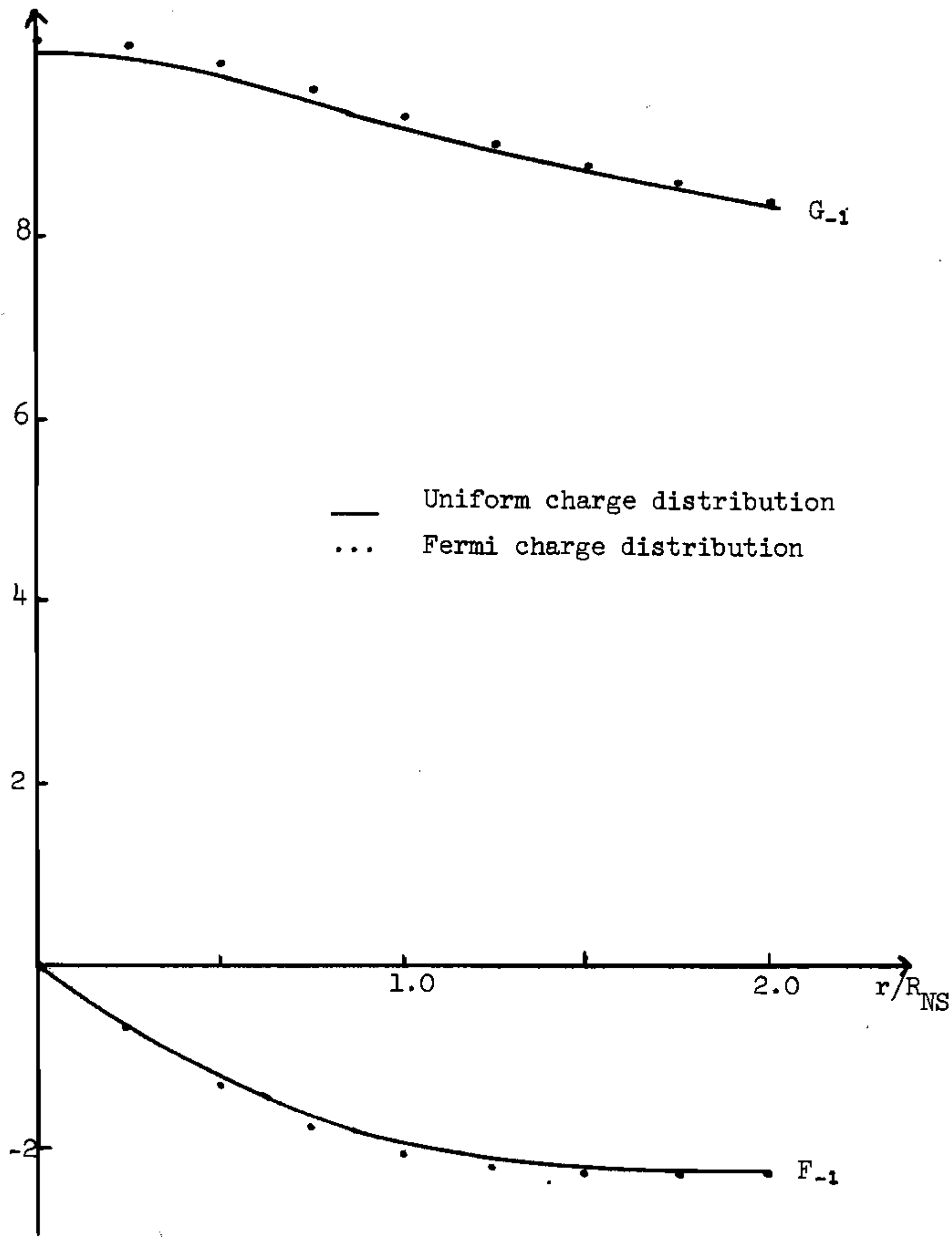


Figure 5. Typical Electron Wave Functions for a Uniform and a Fermi Charge Distribution

For the few electron wave functions which were calculated for this Fermi charge distribution, they were within one percent of those obtained with the uniform charge distribution for the larger in magnitude of F or G, and within four percent for the smaller. A typical case is plotted on Figure 5. So it appears we can obtain the same wave functions as obtained from a Fermi charge distribution by using a uniform distribution and decreasing the value of r_{NS} slightly.

The above results can be summarized in Figure 6. The radial approximation assumes that

$$\frac{A_i(JLKK_\nu)}{r^L} = \left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{r_{NS}}$$

while the two term Buhring approximation assumes that

$$\frac{A_i(JLKK_\nu)}{r^L} = \left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{NS} \frac{1 + \dfrac{{}^1\alpha_i(JLKK_\nu)}{{}^0\alpha_i(JLKK_\nu)}\left(\dfrac{r}{r_{NS}}\right)^2}{\displaystyle\sum_{n=0}^{\infty} \frac{{}^n\alpha_i(JLKK_\nu)}{{}^0\alpha_i(JLKK_\nu)}}$$

Actually, from Figure 6, a two term expansion that better fits the exact curve is an average of these two approximations

$$\frac{A_i(JLKK_\nu)}{r^L} = \frac{1}{2}\left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{NS}\left[1 + \frac{1 + \dfrac{{}^1\alpha_i(JLKK_\nu)}{{}^0\alpha_i(JLKK_\nu)}\left(\dfrac{r}{r_{NS}}\right)^2}{1 + \dfrac{{}^1\alpha_i(JLKK_\nu)}{{}^0\alpha_i(JLKK_\nu)}}\right]$$

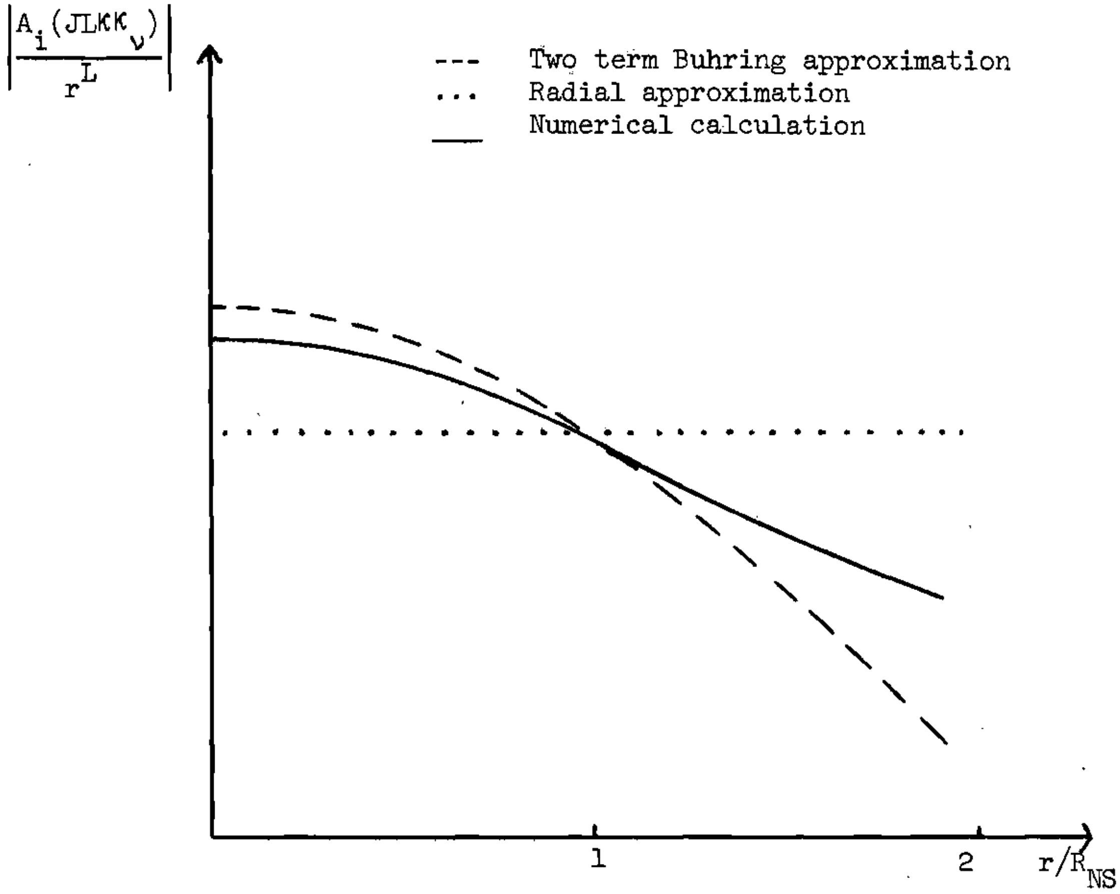


Figure 6. A Sketch of the Lepton Contribution

Keeping in mind that the reason for these approximations is to separate the lepton from the nuclear contribution, let us now reexamine them in light of the above electron wave functions. As Konopinski[5] has indicated, since the lepton contribution does not vary too rapidly in the region where the expected nuclear contribution is non-zero, $0 \leq r < \sim \frac{3}{2} r_{NS}$, then you expect there is a point P such that

$$\int_0^\infty A_i(JL\kappa\kappa_\nu) <O_J^L> r^2 dr = \left[\frac{A_i(JL\kappa\kappa_\nu)}{r^L}\right]_{R_{p_i}(JL\kappa\kappa_\nu)} \int_0^\infty <O_J^L> r^{2+L} dr$$

For the explicit case considered at the end of Chapter I

$$1^- \rightarrow 2^+ + e^- + \bar{\nu}$$

there are 22 terms $\frac{A_i(JL\kappa\kappa_\nu)}{r^L} <O_J^L>$ that appear in the expressions for the beta decay observables. Hence there would be 22 different $R_{p_i}(JL\kappa\kappa_\nu)$'s. These are simply too many parameters to be determined uniquely by experimental data. Therefore, it is assumed

$$R_{p_i}(JL\kappa\kappa_\nu) \simeq R_{p_i}(J'L'\kappa'\kappa_\nu') \simeq r_{NS}$$

hence the radial approximation

$$\int_0^\infty A_i(JL\kappa\kappa_\nu) <O_J^L> r^2 dr \simeq \left[\frac{A_i(JL\kappa\kappa_\nu)}{r^L}\right]_{r_{NS}} \int_0^\infty <O_J^L> r^{2+L} dr$$

However, if there is cancallation

$$\int_0^{\infty} \langle O_J^L \rangle \, r^{a+L} dr = \epsilon \simeq 0$$

then the radial approximation would not be very accurate. This is easily seen from the following case.

Assume $\dfrac{A_i(JLKK_\nu)}{r^L} = 1 - s\left(\dfrac{r}{r_{NS}}\right)^a$ and that there is cancellation $|\epsilon| \ll 1$. For this case, the R_p would be given by

$$\int_0^{\infty} \left[1 - s\left(\frac{r}{r_{NS}}\right)^a\right] \langle O_J^L \rangle \, r^{a+L} dr = \left[1 - s\left(\frac{r_p}{r_{NS}}\right)^a\right] \epsilon$$

If there were complete cancellation, $\epsilon = 0$, and if $s \neq 0$; then r_p would have to be ∞ or $i\infty$. So the closer the nuclear contributions are to cancellation, the worse the approximation becomes. However, if not all of the $\langle O_J^L \rangle$ involved in calculating an observable suffer cancellation, then the observable calculated using the radial approximation will not be too much in error since the terms that suffer cancellation will not contribute significantly.

If, rather than using the radial approximation, we use the Buhring approximation

$$\int_0^{\infty} A_i(JLKK_\nu) \, \langle O_J^L \rangle \, r^a dr = \int_0^{\infty} A_i^{in}(JLKK_\nu) \, \langle O_J^L \rangle \, r^a dr$$

$$+ \int_{r_{NS}}^{\infty} \left[A_i^{out}(JLKK_\nu) - A_i^{in}(JLKK_\nu)\right] \langle O_J^L \rangle \, r^a dr \; .$$

$$\simeq \int_0^\infty A_i^{in}(JLKK_\nu) \langle O_J{}^L \rangle r^2 dr$$

Even if the expression

$$\int_{r_{NS}}^\infty \left[A_i^{out}(JLKK_\nu) - A_i^{in}(JLKK_\nu)\right] \langle O_J{}^L \rangle r^2 dr$$

is equal to zero and a two term expansion is adequate, i.e.,

$$\frac{A_i^{in}(JLKK_\nu)}{r^L} \simeq \frac{1}{2}\left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{r_{NS}} \left[1 + \frac{1 + \frac{{}^1\alpha_i}{{}^0\alpha_i}\left(\frac{r}{r_{NS}}\right)^{2n}}{1 + \frac{{}^1\alpha_i}{{}^0\alpha_i}}\right]$$

we see we have introduced twice as many nuclear contributions as would appear in the radial approximation. For the case $1^- \rightarrow 2^+ + e^- + \bar{\nu}$ using the radial approximation, we have only these terms appearing, $\int \langle Y_1 \rangle r^3 dr$, $\int \langle V_1{}^0 \cdot p \rangle r^2 dr$, $\int \langle V_1{}^1 \cdot \sigma \rangle r^3 dr$, and $\int \langle V_2{}^1 \cdot \sigma \rangle r^3 dr$. However, using the two term Buhring approximation, we would have those as well as the following

$\int \langle Y_1 \rangle r^5 dr$, $\int \langle V_1{}^0 \cdot p \rangle r^4 dr$, $\int \langle V_1{}^1 \cdot \sigma \rangle r^5 dr$, and $\int \langle V_2{}^1 \cdot \sigma \rangle r^5 dr$

This doubling of nuclear matrix elements, even though it should be more accurate, does not simplify the analysis of beta decay data. Even when the radial approximation is used and only four nuclear matrix elements appear in the expressions for beta decay observables, the experimental data are usually not sufficiently accurate to determine uniquely a set of values for the matrix elements.

Another method which might be used to separate the lepton from the nuclear part could be the first mean value theorem. It states: if $\frac{A_i(JLK\kappa_\nu)}{r^L}$ and $\langle O_J{}^L\rangle\, r^{a+L}$ are two continuous functions from 0 to R_E and $\frac{A_i(JLK\kappa_\nu)}{r^L}$ does not change sign in this interval, then there exists at least one value of $R_{p_i}(JLK\kappa_\nu)$ such that

$$0 \leqq R_p \leqq R_E$$

$$\int_0^{R_E} \frac{A_i(JLK\kappa_\nu)}{r^L} \langle O_J{}^L\rangle\, r^{a+L} dr = \Big[\langle O_J{}^L\rangle\, r^{a+L} \Big]_{R_{p_i}(JLK\kappa_\nu)} \int_0^{R_E} \frac{A_i(JLK\kappa_\nu)}{r^L}\, dr$$

Here again for the case of $1^- \rightarrow 2^+ + e^- + \bar{\nu}$, there would be 22 parameters, but worse than that, since $\langle O_J{}^L\rangle\, r^{a+L}$ is oscillatory, there would not necessarily be a unique $R_{p_i}(JLK\kappa_\nu)$ for each case. However, one might make the approximation that for a given J and L

$$R_{p_i}(JLK\kappa_\nu) \simeq R_{p_i}(JLK'\kappa_\nu')$$

which would then reduce the expressions to four parameters. This probably is not any better than the radial approximation and, in addition, would require extensive numerical tabulations of

$$\int_0^{R_E} \frac{A_i(JLK\kappa_\nu)}{r^L}\, dr$$

One could also start with the second mean value theorem to attempt to find a convenient way to separate the lepton from the nuclear part. It states that if $A_i(JLKK_\nu)/r^L$ is a positive monotonic decreasing function, then there exists an $R_{p_i}(JLKK_\nu)$

$$0 \leq R_{p_i}(JLKK_\nu) \leq R_E$$

such that

$$\int_0^{R_E} \frac{A_i(JLKK_\nu)}{r^L} \langle O_J{}^L \rangle \, r^{a+L} dr = \left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{r=o} \int_0^{R_{p_i}(JLKK_\nu)} \langle O_J{}^L \rangle \, r^{a+L} dr$$

Here again there would be 22 $R_{p_i}(JLKK_\nu)$'s but at least $\left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{r=o}$ would be easy to evaluate by using the Bhalla and Rose tables and

$$\left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{r=o} = \left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{NS} \sum_{n=o}^{\infty} \frac{{}^n\alpha_i(JLKK_\nu)}{{}^o\alpha_i(JLKK_\nu)}$$

Here again you might make the approximation that for a given J and L

$$R_{p_i}(JLKK_\nu) \simeq R_{p_i}(JLK'K_\nu')$$

and again reduce the expression to four parameters, but due to the oscillatory nature of the nuclear contribution, the integral

$$\int_0^{R_p} \langle O_J{}^L \rangle \, r^{a+L} dr$$

might be very sensitive to R_p.

From the above arguments, it appears that the easiest way to improve the radial approximation is by using a Buhring two term approximation. This would double the number of nuclear matrix elements appearing as parameters in the expressions for beta decay observables which are to be fitted by experiment. This is probably too many to give useful results for data analysis.

We see that, no matter what approximation we use to separate the lepton from the nucleon part of the radial integral, the approximation depends on the nuclear wave functions. Hence, when we use experimental data to evaluate the nuclear contribution, we are simultaneously limiting ourselves to the type of wave functions, and hence nuclear models, for which the approximation works.

CHAPTER III

THE NILSSON TWO PARTICLE MATRIX ELEMENTS

In this chapter, the necessary formalism will be presented in order to evaluate the beta decay matrix elements using the Nilsson model. As will be shown, this formalism can also be used for the Faessler and Sheline wave functions obtained by using a deformed Woods-Saxton potential.

The single particle Nilsson wave function, which has rotational symmetry about the nuclear axis, reflection symmetry about a plane perpendicular to this axis, a definite parity, and is in the vibrational ground state, can be written as[3]

$$U = \left(\frac{2I+1}{16\pi^2}\right)^{\frac{1}{2}} \left[\psi_\Omega D^I_{MK} + (-)^{I-j} \psi_{-\Omega} D^I_{M-K}\right] ; K = \Omega$$

Nilsson has tabulated the a's appearing in the expression for the intrinsic wave function

$$\psi_\Omega = \sum a_{\ell\Lambda}(\Omega)\, R_{N\ell}(\rho)\, Y_{\ell\Lambda}\, \chi_\Sigma$$

$$\ell = N, N-2, \ldots 1 \text{ or } 0$$

$$\Sigma = \pm \tfrac{1}{2}$$

$$\Lambda\,(\Lambda + \Sigma = \Omega) = \ell, \ell-1, \ldots, -\ell$$

where $\rho = a\, r = \left(\frac{M\, \omega_0(\delta)}{\hbar}\right)^{\frac{1}{2}} r.$

$$\omega_0 = \overset{\circ}{\omega}_0/(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3)^{1/6} = \eta\kappa\,\overset{\circ}{\omega}_0/\delta\ ; \quad \hbar\,\overset{\circ}{\omega}_0 = 41/A^{\frac{1}{3}}\ \text{MeV}$$

Hence the parity of ψ_Ω is $(-)^N$ and because of three-axis symmetry (reference 18, p. 258)

$$a_{\ell\Lambda}(\Omega) = a_{\ell-\Lambda}(-\Omega)$$

and

$$(-)^j\ \psi_{-\Omega} = (-)^{N-\frac{1}{2}}\ \psi_{-\Omega}$$

The radial part is given by

$$R_{N\ell} \equiv C_{n\ell}\ \rho^{\ell}\ e^{-\ell^2/2}\ F(-n,\ \ell + \frac{3}{2},\ \rho^2); \quad n \equiv \frac{N - \ell}{2}$$

where $F(-n,\ \ell + \frac{3}{2},\ \rho^2)$ is the confluent hypergeometric function, i.e.,

$$F(a,c,x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

$Y_{\ell\Lambda}$ is the spherical harmonic and χ_Σ is the spin one-half wave function. The normalization constant $C_{n\ell}$ is determined from

$$\int |R_{N\ell}|^2\ \rho^2\ d\rho = 1$$

and its phase is fixed by making it satisfy Nilsson's integral expression.

Therefore $$C_{n\ell} = \frac{(-)^n}{\Gamma\left(\ell + \frac{3}{2}\right)}\left(\frac{2\Gamma\ (n + \ell + \frac{3}{2})}{n!}\right)^{\frac{1}{2}}$$

The integrals encountered are gamma functions, i.e.,

$$\int_0^\infty \rho_1{}^n \, e^{-\rho_1{}^2/2} \, \rho_2{}^m \, e^{-\rho_2{}^2/2} \, r^{L+2} \, dr$$

$$= \frac{a_1{}^n \, a_2{}^m}{2\left(\frac{a_1{}^2 + a_2{}^2}{2}\right)^{\frac{n+m+L+3}{2}}} \, \Gamma\left(\frac{n + m + L + 3}{2}\right)$$

For $a_1 = a_2$ this reduces to

$$\frac{1}{2a_1{}^{L+3}} \, \Gamma\left(\frac{n + m + L + 3}{2}\right)$$

Faessler and Sheline[19] have tabulated the C's in their expression for their intrinsic wave functions which can be rewritten as

$$\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} \sum_{\substack{j=\ell\pm\frac{1}{2} \\ \ell=N,N-2,\ \ldots\ 1 \text{ or } 0 \\ \Sigma=\pm\frac{1}{2} \\ \Lambda=\ell,\ell-1,\ \ldots,\ -\ell \\ \Lambda+\Sigma=\Omega}} C_j \, (-)^{\frac{1}{2}-\ell-\Omega} \begin{pmatrix} \ell & \frac{1}{2} & j \\ \Lambda & \Sigma & -\Omega \end{pmatrix} R_{n\ell} \, Y_{\ell\Lambda} \, \chi_\Sigma$$

Hence the Nilsson a's can be written in terms of the Faessler and Sheline C's by

Table 5. Some Nilsson Normalization Constants and Radial Functions

$$C_{05} = \left(\frac{2}{\Gamma(13/2)}\right)^{\frac{1}{2}} \qquad C_{04} = \left(\frac{2}{\Gamma(11/2)}\right)^{\frac{1}{2}}$$

$$C_{13} = -\left(\frac{9}{\Gamma(9/2)}\right)^{\frac{1}{2}} \qquad C_{12} = -\left(\frac{7}{\Gamma(7/2)}\right)^{\frac{1}{2}}$$

$$C_{21} = \left(\frac{35}{4\Gamma(5/2)}\right)^{\frac{1}{2}} \qquad C_{20} = \left(\frac{15}{4\Gamma(3/2)}\right)^{\frac{1}{2}}$$

$$R_{55} = C_{05}\, e^{-\rho^2/2}\, \rho^5$$

$$R_{53} = C_{13}\, e^{-\rho^2/2}\, (\rho^3 - \frac{2}{9}\rho^5)$$

$$R_{51} = C_{21}\, e^{-\rho^2/2}\, (\rho - \frac{4}{5}\rho^3 + \frac{4}{35}\rho^5)$$

$$R_{44} = C_{04}\, e^{-\rho^2/2}\, \rho^4$$

$$R_{42} = C_{12}\, e^{-\rho^2/2}\, (\rho^2 - \frac{2}{7}\rho^4)$$

$$R_{40} = C_{20}\, e^{-\rho^2/2}\, (1 - \frac{4}{5}\rho^2 + \frac{4}{15}\rho^4)$$

$$a_{\ell\Lambda} = \left(\frac{m\omega}{h}\right)^{\frac{1}{2}} (-)^{\frac{1}{2}-\ell-\Omega} \left[C_{\ell+\frac{1}{2}} \begin{pmatrix} \ell & \frac{1}{2} & \ell+\frac{1}{2} \\ \Lambda & \Sigma & -\Omega \end{pmatrix} + C_{\ell-\frac{1}{2}} \begin{pmatrix} \ell & \frac{1}{2} & \ell-\frac{1}{2} \\ \Lambda & \Sigma & -\Omega \end{pmatrix}\right]$$

Thus by using this expression, the following presentation can be used for the Faessler and Sheline wave functions.

In order to treat an even mass nucleus, it is assumed[20] that the intrinsic nuclear state can be represented using the Nilsson wave functions for the last two nucleons. Here, following Tuong, et al.,[1] the Nilsson two particle matrix elements will be presented for $pn \rightarrow pp'$.

The initial intrinsic wave function used is

$$\psi_{\Omega_i}(pn) = \psi^{(1)}_{N_p\Omega_p} (\eta'\ \mu'\ \delta')\ \psi^{(2)}_{N_n\Omega_n} (\eta_n\ \mu_n\ \delta_n)$$

and the final is

$$\psi_{\Omega}(pp') = \frac{1}{\sqrt{2}} \left[\psi^{(1)}_{N_p\Omega_p'} (\eta\ \mu\ \delta)\ \psi^{(2)}_{N_p\Omega_p'} (\eta\ \mu\ \delta)\right.$$

$$\left. -\ \psi^{(1)}_{N_p\Omega_p'} (\eta\ \mu\ \delta) \qquad \psi^{(2)}_{N_p\Omega_p} (\eta\ \mu\ \delta)\right]$$

η, μ, and δ are the dependent set of Nilsson model parameters. Tuong, et al., have presented the formalism for

$$\eta' = \eta_n = \eta$$

$$\mu' = \mu_n = \mu$$

$$\delta' = \delta_n = \delta$$

Since it seems reasonable that the nucleons see a different average potential after the neutron has decayed into a proton, here it will not be assumed that the parameters are the same.

For the particular case $\Omega_{p'} = -\Omega_p$, i.e., K = 0, and for I being an even number and using that

$$\int \psi_{N\Omega}^{*}(a)\ \psi_{N-\Omega}(a')\ d\Omega = 0$$

then Tuong's reduced matrix element can be written as

$$(I\|O_{JL}\|I_i) = b\left(\frac{2(2I_i+1)}{2I+1}\right)^{\frac{1}{2}} (I_i\ J\ I : K_i\ -K\ 0)$$

$$\int \psi_{N_p -\Omega_p}(\eta\ \mu\ \delta)^{*}\ O_{JL}^{-K_i}\ \psi_{N_n\Omega_n}(\eta_n\ \mu_n\ \delta_n)\ d\Omega$$

where

$$b \equiv \int \psi_{N_p\Omega_p}(\eta\ \mu\ \delta)\ \psi_{N_p\Omega_p}(\eta'\ \mu'\ \delta')\ d\Omega$$

The relations between Tuong's reduced matrix ~~elements and~~ those used in this paper are found in Table 1.

If $\eta = \eta'$, $\mu = \mu'$, and $\delta = \delta'$, then $\int b\ \rho^2\ d\rho = 1$, which is Tuong's result.

However, if the initial and final parameters are not the same, e.g.

$$|4\tfrac{1}{2}\ 43\rangle = [411\downarrow] \qquad \rightarrow [411\downarrow]$$

$$\begin{array}{ll} \mu' = 0.55 & \mu = 0.55 \\ \eta' = 6 & \eta = 4 \\ \kappa' = 0.05 & \kappa = 0.05 \end{array}$$

then for this particular case

$$\int b\, \rho'^2\, d\,\rho' = a'^3 \int r^2 dr\ \{R_{40}(ar)\, R_{40}(a'r)\, a_{00}\, a_{00}'$$

$$+ R_{42}(ar)\, R_{42}(a'r)\, [a_{21}a_{21}' + a_{20}a_{20}']$$

$$+ R_{44}(ar)\, R_{44}(a'r)\, [a_{41}a_{41}' + a_{40}a_{40}']\}$$

where $\rho = ar = \left(\frac{M\omega_0(\delta)}{\hbar}\right)^{\frac{1}{2}} r = \left(\frac{M}{\hbar}\, \frac{\overset{\circ}{\omega}_0\ \eta\ \kappa}{\delta}\right)^{\frac{1}{2}} r = \left(\frac{M}{\hbar}\, \frac{\overset{\circ}{\omega}_0}{\left(1-\frac{4}{3}\,\delta^2 - \frac{16}{27}\,\delta^3\right)^{1/6}}\right)^{\frac{1}{2}} r$

and $a_{ij}' = a_{ij}(\eta'\ \mu'\ \delta')$. Using

$$B \equiv \left(\frac{a}{a'}\right)^2 \quad \text{and} \quad A \equiv \frac{1+B}{2}$$

then it can be rewritten after integration as

$$\int b\, \rho'^2\, d\,\rho' = \frac{1}{2}\left\{\frac{15}{4}\, a_{00}a_{00}' \left[\frac{1}{A^{3/2}} - 2\,\frac{B+1}{A^{5/2}} + \frac{1+B^2}{A^{7/2}}\right]\right.$$

$$+ \frac{B}{A^{7/2}}\left[(a_{21}a_{21}' + a_{20}a_{20}')(7) + 25\, a_{00}a_{00}'\right]$$

$$+ 7\,\frac{B+B^2}{A^{9/2}}\left[-\frac{5}{2}\, a_{00}a_{00}' - (a_{21}a_{21}' + a_{20}a_{20}')\right]$$

$$+ \frac{B^2}{A^{11/2}}\left[\frac{63}{4}\, a_{00}a_{00}' + 9(a_{21}a_{21}' + a_{20}a_{20}')\right.$$

$$\left.\left. + 2(a_{41}a_{41}' + a_{40}a_{40}')\right]\right\}$$

Using Nilsson's Figure 5 and Table 1-b,[3] B = .99 and the a's are

for $\eta = 4$	and	for $\eta = 6$
0.176	a_{40}	0.163
- 0.123	a_{20}	- 0.062
- 0.343	a_{00}	- 0.279
- 0.373	a_{41}	- 0.445
0.834	a_{21}	0.833

$A^{\frac{1}{2}} = \sqrt{0.99855}$ $\quad\quad$ $\int b\, \rho'^2\, d\, \rho' = 0.9965$

Hence for this case, the matrix elements would be lowered by only 0.35 percent.

However, let us look at a nucleus containing 170 nucleons and, for this argument, assume they are distinguishable. Hence the intrinsic wave function for the system is a product of the 170 Nilsson wave functions

$$\Pi_{i=1}^{170}\ \psi_i$$

If the 170th nucleon undergoes a transition described by an operator O_{170}, then the matrix element for this transition would be

$$\psi_{170}'^{*}\ O_{170}\ \psi_{170}\ \prod_{i=1}^{169}\ \psi_i'^{*}\ \psi_i$$

where the primes represent the final state wave functions.

If the final state parameters were different from the first and were such that all 169

$$\int \psi_i'^{*} \psi_i \, d\tau = 0.9965$$

then the matrix element would be proportional to

$$(0.9965)^{169} \psi_{170}'^{*} O_{170} \psi_{170}$$

$$= 0.58 \psi_{170}'^{*} O_{170} \psi_{170}$$

However, if the final parameters were such that all 169

$$\int \psi_i^{*} \psi_i \, d \, d\tau = 0.965$$

then the matrix element would be proportional to

$$(0.965)^{169} \psi_{170}'^{*} O_{170} \psi_{170}$$

$$= 0.003 \psi_{170}' O_{170} \psi_{170}$$

The normalized shape and A_2 coefficient depend on the ratios of matrix elements so that this change in the final state parameters of the non-interacting nucleons plays no role. However, in calculating the log ft value for the last case considered, you would introduce a factor of 10^5 to the ft value or increase the log ft value by plus five.

The Nilsson single particle radial matrix elements, i.e.,

$$\int \psi_{N_p \Omega_p}^{*} (\eta \mu \delta) O_{JL}^{-K_i} \psi_{N_n \Omega_n}(\eta_n \mu_n \delta_n) \, d\Omega$$

$$= \sum_{\substack{\ell'=N_p, N_p-2, \ldots, 1 \text{ or } 0 \\ \ell, \Sigma, \Sigma' \\ \Lambda, \Lambda' (\Lambda+\Sigma=\Omega)}} \delta_{\Sigma, \pm\frac{1}{2}} \delta_{\Sigma, \pm\frac{1}{2}} a_{\ell'\Lambda'} (-\Omega_p, \eta, \mu) \, a_{\ell\Lambda} (\Omega_n, \eta_n, \mu_n)$$

$$R_{N_p \ell'} (\delta) \, R_{N_n \ell} (\delta_n) \int Y_{\ell'\Lambda'}^{*} \chi_{\Sigma'}^{*} \, O_{JL}^{-K_i} \chi_{\Sigma} Y_{\ell\Lambda} d\Omega$$

can be written in the form of Bogdan and Lipnick[20] if we integrate it over $\int_0^{\infty} r^2 dr$. This yields

$$a_p(\eta \, \mu)^* \, T_{JL}^{-K_i} \, a_n(\eta_n \, \mu_n)$$

where the a's are column vectors formed by the $a_{\ell\Lambda}$'s and the matrix $T_{JL}^{-K_i}$ is independent of the $a_{\ell\Lambda}$'s. Bogdan and Lipnick have indicated that the T matrix is independent of η, μ, η_n, and μ_n because they have chosen η_n and μ_n such that $\delta_n = \delta$, i.e., $\rho_n = \rho$. Thus when $\delta_i = \delta_f$

$$\prod_{i=1}^{A-1} \int \psi_f^* \, \psi_i \, d\tau = 1$$

and we do not have to worry about the non-transforming nucleons contributing to the ft value. However, you cannot use the tables for $a_{\ell\Lambda}$ in Nilsson or in Mottelson and Nilsson[3] because for $\eta_i \neq \eta_f$, $\delta_i \neq \delta_f$. One would have to calculate the $a_{\ell\Lambda}$'s as Bogdan and Lipnick have done.

Using Tuong's expressions and realizing that

$$<\chi^{+}_{\Omega_f} | T_{JLM} (\hat{r}, \vec{A}) | \chi_{\Omega_i} > = \int (\chi_{N\Omega} | T_{JLM} (\hat{r}, \vec{A}) | \chi_{N_i \Omega_i}) \; r^{2+L} dr$$

the necessary Nilsson single particle matrix elements are listed in Table 6, according to Tuong's notation. In Table 7, the Nilsson wave functions in our notation are tabulated, and in Table 8 the various R combinations are explicitly written out.

Shown in Figure 7 are typical plots of the radial nuclear matrix elements which are defined in the following way.

$$i\, C_V \langle Y_1 \rangle\, r^3 = \left(\frac{2a^3}{4\pi}\right)^{\frac{1}{2}} C_V\, M_R$$

$$C \langle V_1{}^0 \cdot p \rangle\, r^3 = -\, C_V\, i \left(\frac{a^3}{3(4\pi)}\right)^{\frac{1}{2}} M_A$$

$$-\, i\, C_A \langle V_1{}^1 \cdot \sigma \rangle\, r^3 = -\, C_A \left(\frac{a^3}{4\pi}\right)^{\frac{1}{2}} M_K$$

$$i\, C_A \langle V_2{}^1 \cdot \sigma \rangle\, r^3 = C_A \left(\frac{3a^3}{4\pi}\right)^{\frac{1}{2}} M_B$$

They are plotted for the electron decay of ${}_{69}Tm^{170}$ into ${}_{70}Yb^{170}$. The dashed curve corresponds to the Nilsson states suggested by Gallagher and Solovier,[20] where 101 neutron is described by the Nilsson state

$$\tfrac{1}{2} - [521\downarrow] = |5\tfrac{1}{2}\, 63\rangle$$

and transforms into the proton state

$$\tfrac{1}{2} + [411\downarrow] = |4\tfrac{1}{2}\, 43\rangle$$

which is the same Nilsson function for the 69^{th} proton before and after transition. The Nilsson a coefficients which were used are listed on page 72.

Table 6. The Nilsson Single Particle Matrix Elements

$$(\psi_{N\Omega}|Y_1^{\,M}|\,\psi_{N_i\Omega_i}) = -\left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \sum_{\substack{\ell_i=N_i,N_i-2,\,---,\,1\text{ or }0\\ \ell=N,N-2,\,---,\,1\text{ or }0\\ \Sigma=\pm\frac{1}{2}}} (-)^{\Sigma-\Omega-\ell_i} \begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\Sigma & M & \Sigma-\Omega \end{pmatrix}$$

$$a_{\ell_i,\Omega_i-\Sigma}\, a_{\ell,\Omega-\Sigma} \left(\delta_{\ell,\ell_i+1}\sqrt{\ell_i+1} - \delta_{\ell,\ell_i-1}\sqrt{\ell_i}\right)$$

$$R_{N\ell}\, R_{N_i\ell_i}$$

$$(\psi_{N\Omega}|T^{M}_{J0}\,(\hat{r}\,\vec{p})|\psi_{N_i\Omega_i}) = i\,\delta_{J1}\,\frac{\sqrt{3}}{4\pi}\sum_{\ell,\ell_i\Sigma} (-)^{\Sigma-\Omega-\ell_i} \begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\Sigma & M & \Sigma-\Omega \end{pmatrix} a_{\ell_i,\Omega_i-\Sigma}$$

$$a_{\ell,\Omega-\Sigma}\, R_{N\ell}\left(\delta_{\ell,\ell_i+1}\sqrt{\ell_i+1}\, D_-\,(\ell_i)\right.$$

$$\left.-\,\delta_{\ell,\ell_i-1}\sqrt{\ell_i}\, D_+\,(\ell_i)\right) R_{N_i\ell_i}$$

$$(\psi_{N\Omega}|T^{M}_{J1}\,(\hat{r}\,\vec{\sigma})|\psi_{N_i\Omega_i}) = \frac{3\sqrt{2J+1}}{4\pi}\sum_{\ell,\ell_i,\Sigma} (-)^{\Sigma-\Omega-\ell_i+M+1}\, a_{\ell_i,\Omega_i-\Sigma_i}\, a_{\ell,\Omega-\Sigma}$$

$$(\delta_{\ell,\ell_i+1}\sqrt{2\ell_i+1} - \delta_{\ell,\ell_i-1}\sqrt{\ell_i})\, R_{N\ell}\, R_{N_i\ell_i}$$

$$\left\{\delta_{\Sigma\Sigma_i}\,(-)^{\Sigma_i-\frac{1}{2}}\begin{pmatrix} 1 & 1 & J \\ -M & 0 & M \end{pmatrix}\begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\Sigma & M & \Sigma-\Omega \end{pmatrix}\right.$$

$$+\,\delta_{\Sigma_i\,\frac{1}{2}}\,\delta_{\Sigma-\frac{1}{2}}\sqrt{2}\begin{pmatrix} 1 & 1 & J \\ -1-M & 1 & M \end{pmatrix}\begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\frac{1}{2} & 1+M & -\frac{1}{2}-\Omega \end{pmatrix}$$

$$\left.-\,\delta_{\Sigma_i-\frac{1}{2}}\,\delta_{\Sigma\,\frac{1}{2}}\sqrt{2}\begin{pmatrix} 1 & 1 & J \\ 1-M & -1 & M \end{pmatrix}\begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i+\frac{1}{2} & M-1 & \frac{1}{2}-\Omega \end{pmatrix}\right\}$$

Table 7. The Nuclear Matrix Elements for First Forbidden Beta Decay

$$\langle Y_1 \rangle = i\, b \left(\frac{6}{4\pi}\right)^{\frac{1}{2}} (-)^{I-I_i} (I_i\, 1 I | K_i - K_i 0) \sum_{\ell_i \ell \Sigma} (-)^{\Sigma-\Omega-\ell_i}\, a_{\ell_i, \Omega_i-\Sigma}$$

$$a_{\ell,\Omega-\Sigma} \begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\Sigma & -K_i & \Sigma-\Omega \end{pmatrix} \left\{ \delta_{\ell,\ell_i+1} \sqrt{\ell_i+1} - \delta_{\ell,\ell_i-1} \sqrt{\ell_i} \right\}$$

$$R_{N\ell}\, R_{N_i \ell_i}$$

$$\langle V_1{}^{\circ} \cdot p \rangle = i\, b \left(\frac{2}{4\pi}\right)^{\frac{1}{2}} (-)^{I-I_i} (I_i\, 1 I | K_i - K_i 0) \sum_{\ell_i \ell \Sigma} (-)^{\Sigma-\Omega-\ell_i}\, a_{\ell_i, \Omega_i-\Sigma}$$

$$a_{\ell,\Omega-\Sigma} \begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\Sigma & -K_i & \Sigma-\Omega \end{pmatrix} R_{N\ell} \left\{ \delta_{\ell,\ell_i+1} \sqrt{\ell_i+1}\, D_-(\ell_i) - \delta_{\ell,\ell_i-1} \right.$$

$$\left. \sqrt{\ell_i}\, D_+(\ell_i) \right\} R_{N_i \ell_i}$$

$$\langle V_J{}^1 \cdot \sigma \rangle_{J=1,2} = i\, b \left(\frac{6}{4\pi}(2J+1)\right)^{\frac{1}{2}} (-)^{I-I_i} (I_i\, 1 I | K_i - K_i 0) \sum_{\ell \ell_i \Sigma \Sigma_i}$$

$$(-)^{\Sigma-\Omega-\ell_i-K_i}\, a_{\ell_i, \Omega_i-\Sigma_i}\, a_{\ell,\Omega-\Sigma} \left\{ \delta_{\Sigma\Sigma_i} (-)^{\Sigma_i-\frac{1}{2}} \begin{pmatrix} 1 & 1 & J \\ K_i & 0 & -K_i \end{pmatrix} \right.$$

$$\begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\Sigma_i & -K_i & \Sigma-\Omega \end{pmatrix} - \sqrt{2}\, \delta_{\Sigma_i \frac{1}{2}}\, \delta_{\Sigma_f -\frac{1}{2}} \begin{pmatrix} 1 & 1 & J \\ 1+K_i & -1 & -K_i \end{pmatrix}$$

$$\left. \begin{pmatrix} \ell_i & 1 & \ell \\ \Omega_i-\Sigma_i & -K_i-1 & \Sigma-\Omega \end{pmatrix} \right\} \left\{ \delta_{\ell,\ell_i+1} \sqrt{2\ell_i+1} - \delta_{\ell,\ell_i-1} \sqrt{\ell_i} \right\}$$

$$R_{N\ell}\, R_{N_i \ell_i}$$

Table 8. The Nilsson Radial Contributions

$$R_{44}(\rho_2)\, R_{55}(\rho_1) = \frac{2}{\Gamma(11/2)} \sqrt{2/11}\ a_1{}^9\ B^2\ e^{-Aa_1{}^2 r^2}\ r^9$$

$$R_{44}(\rho_2)\, D_+\, R_{55}(\rho_1) = \frac{2}{\Gamma(11/2)} \sqrt{2/11}\ a_1{}^9\ B^2\ e^{-Aa_1{}^2 r^2} (11r^8 - a^2\ r^{10})$$

$$R_{44}(\rho_2)\, R_{53}(\rho_1) = \frac{-2}{\Gamma(9/2)}\ a_1{}^7\ B^2\ e^{-Aa_1{}^2 r^2} (r^7 - \frac{2}{9}\ a_1{}^2\ r^8)$$

$$R_{44}(\rho_2)\, D_-\, R_{53}(\rho_1) = \frac{2/9}{\Gamma(9/2)}\ a_1{}^9\ B^2\ e^{-Aa_1{}^2 r^2}\ (13r^8 - 2a_1{}^2\ r^{10})$$

$$R_{42}(\rho_2)\, R_{53}(\rho_1) = \frac{\sqrt{18}}{\Gamma(7/2)}\ a_1{}^5\ B\ e^{-Aa_1{}^2 r^2}\ (r^5 - a_1{}^2\ r^7\ [\frac{2B}{7} + \frac{2}{9}]$$

$$+ \frac{4}{63}\ a_1{}^4\ r^9\ B)$$

$$R_{42}(\rho_2)\, D_+\, R_{53}(\rho_1) = \frac{\sqrt{18}}{\Gamma(7/2)}\ a_1{}^5\ B\ e^{-Aa_1{}^2 r^2}\ (7r^4 - a_1{}^2\ r^6\ [3 + 2B]$$

$$+ a_1{}^4\ r^8\ [\frac{2}{9} + \frac{6}{7}\ B] - \frac{4}{63}\ a_1{}^6\ B\ r^{10})$$

$$R_{42}(\rho_2)\, R_{51}(\rho_1) = \frac{-7}{\sqrt{2}} \frac{a_1{}^3\ B}{\Gamma(5/2)}\ e^{-Aa_1{}^2 r^2}\ (r^3 - 2a_1{}^2\ r^5\ [\frac{B}{7} + \frac{2}{5}]$$

$$+ \frac{4}{35}\ a_1{}^4\ r^7\ [2B + 1] - \frac{8}{245}\ a_1{}^6\ B\ r^9)$$

$$R_{42}(\rho_2)\, D_-\, R_{51}(\rho_1) = \frac{-7}{\sqrt{2}} \frac{a_1{}^3\ B}{\Gamma(5/2)}\ e^{-Aa_1{}^2 r^2}\ (- \frac{13}{5}\ a_1{}^2\ r^4 + \frac{a_1{}^4\ r^6}{35}\ [44 + 36\ B]$$

$$- \frac{4}{35}\ a_1{}^6\ r^8\ [1 + \frac{22}{7}\ B] + \frac{8}{5(49)}\ a_1{}^8\ B\ r^{10})$$

Table 8. The Nilsson Radial Contributions (Concluded)

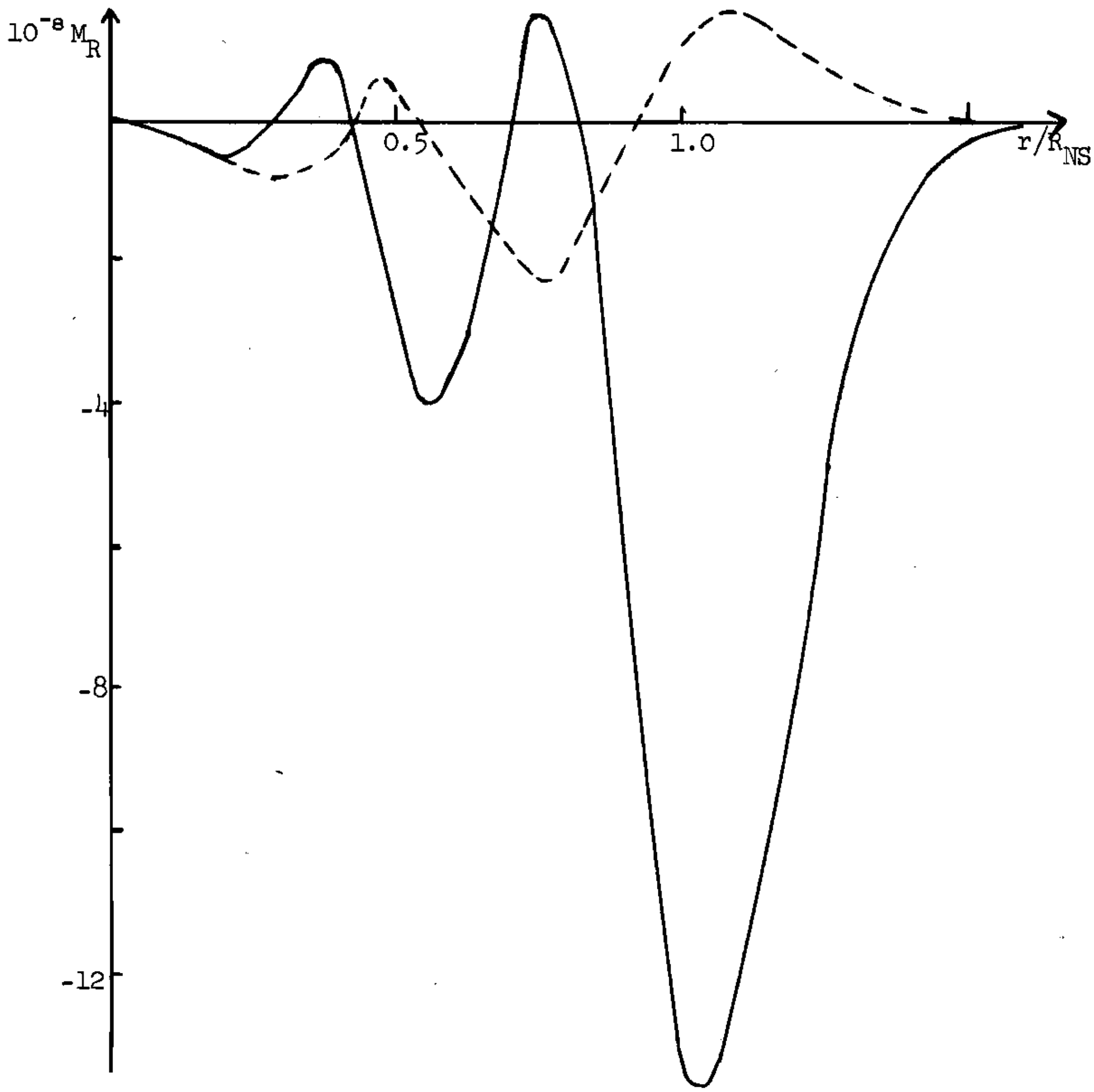
$$R_{40}(\rho_2)\,R_{51}(\rho_1) = \frac{5}{4}\frac{\sqrt{14}}{\Gamma(3/2)}\,a_1\,e^{-Aa_1^2r^2}\,(r - 4a_1^2\,r^3\,[\frac{1}{5} + \frac{B}{3}] + \frac{4}{5}\,a_1^4\,r^5$$

$$[\frac{1}{7} + \frac{4}{3}\,B + \frac{B^2}{3}] - \frac{16}{15}\,a_1^6\,r^7\,[\frac{B}{7} + \frac{B^2}{5}] + \frac{16}{525}\,a_1^8\,r^9\,B^2)$$

$$R_{40}(\rho_2)\,D_+\,R_{51}(\rho_1) = \frac{5}{4}\frac{5\sqrt{14}}{4\Gamma(3/2)}\,a_1\,e^{-Aa_1^2r^2}\,(3 - a_1^2\,r^2\,[5 + 4B]$$

$$+ \frac{4}{15}\,a_1^4\,r^4\,[6 + 25B + 3B^2] - \frac{4}{5}\,a_1^6\,r^6\,[\frac{1}{7} + \frac{8}{3}\,B + \frac{5}{3}\,B^2]$$

$$+ \frac{16}{15}\,a_1^8\,r^8\,[\frac{B}{7} + \frac{2}{5}\,B^2] - \frac{16}{525}\,a_1^{10}\,B^2\,r^{10})$$



—— Faessler and Sheline: $\frac{1}{2}\,[501]_n \rightarrow -\frac{1}{2}\,[41\text{-}1]_p$

--- Nilsson: $|5\tfrac{1}{2}\ 63> \rightarrow |4\tfrac{1}{2}\ 43>$; $\eta = 0.6$

$\mu = 0.45$ $\mu = 0.55$

Figure 7. The Radial Nuclear Matrix Elements for the Decay of Tm^{170}

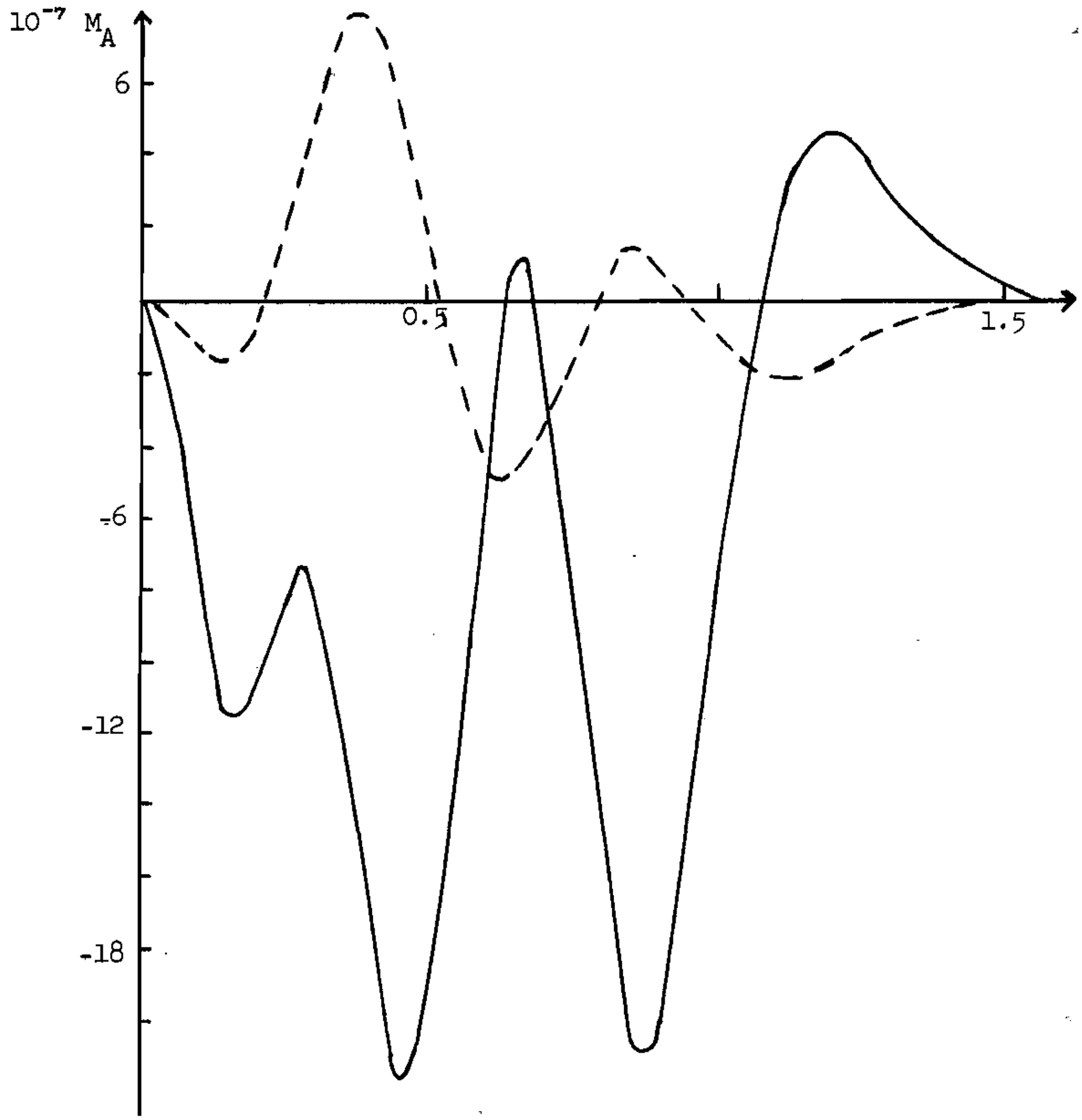


Figure 7. Continued

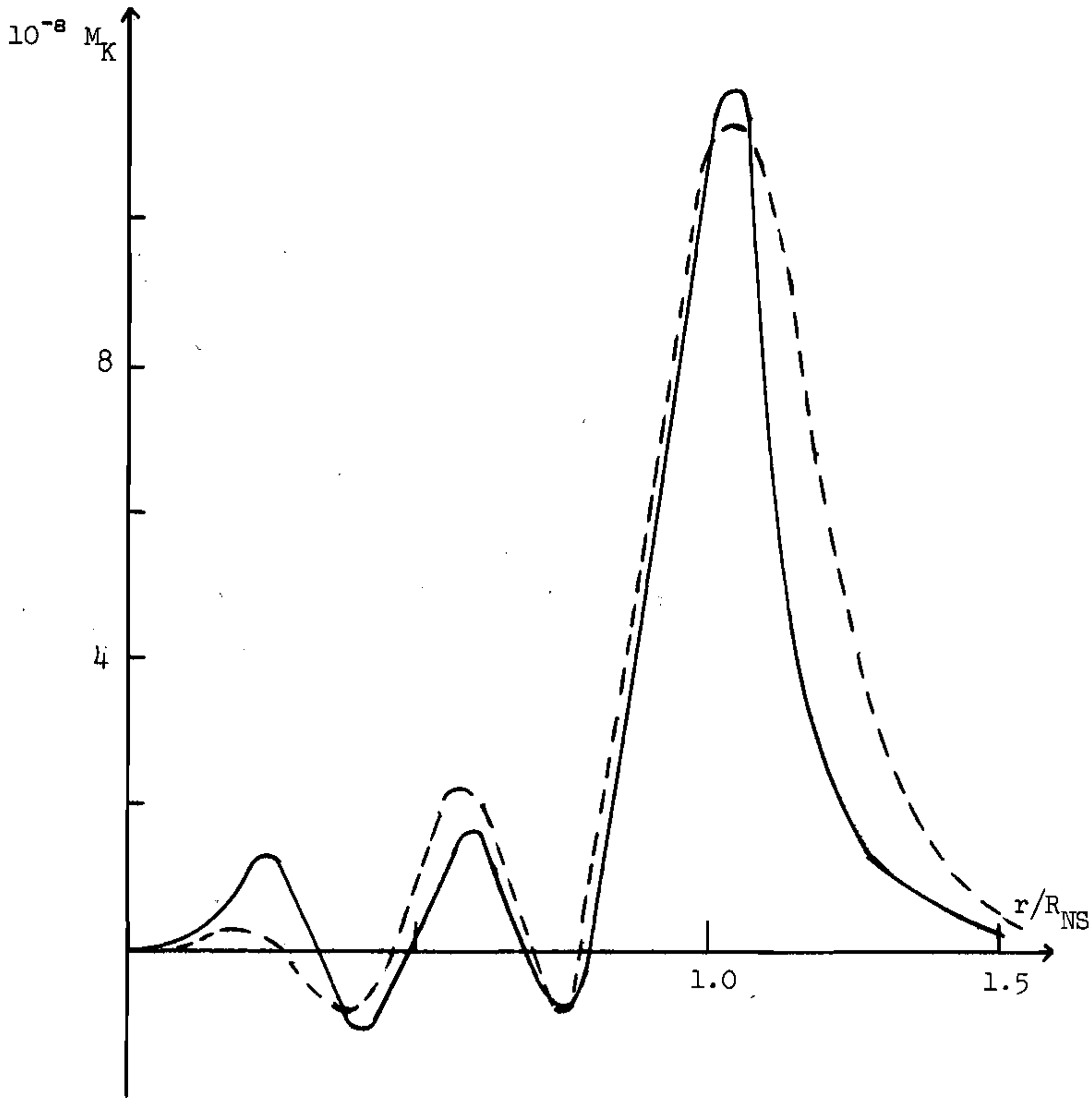


Figure 7. Continued

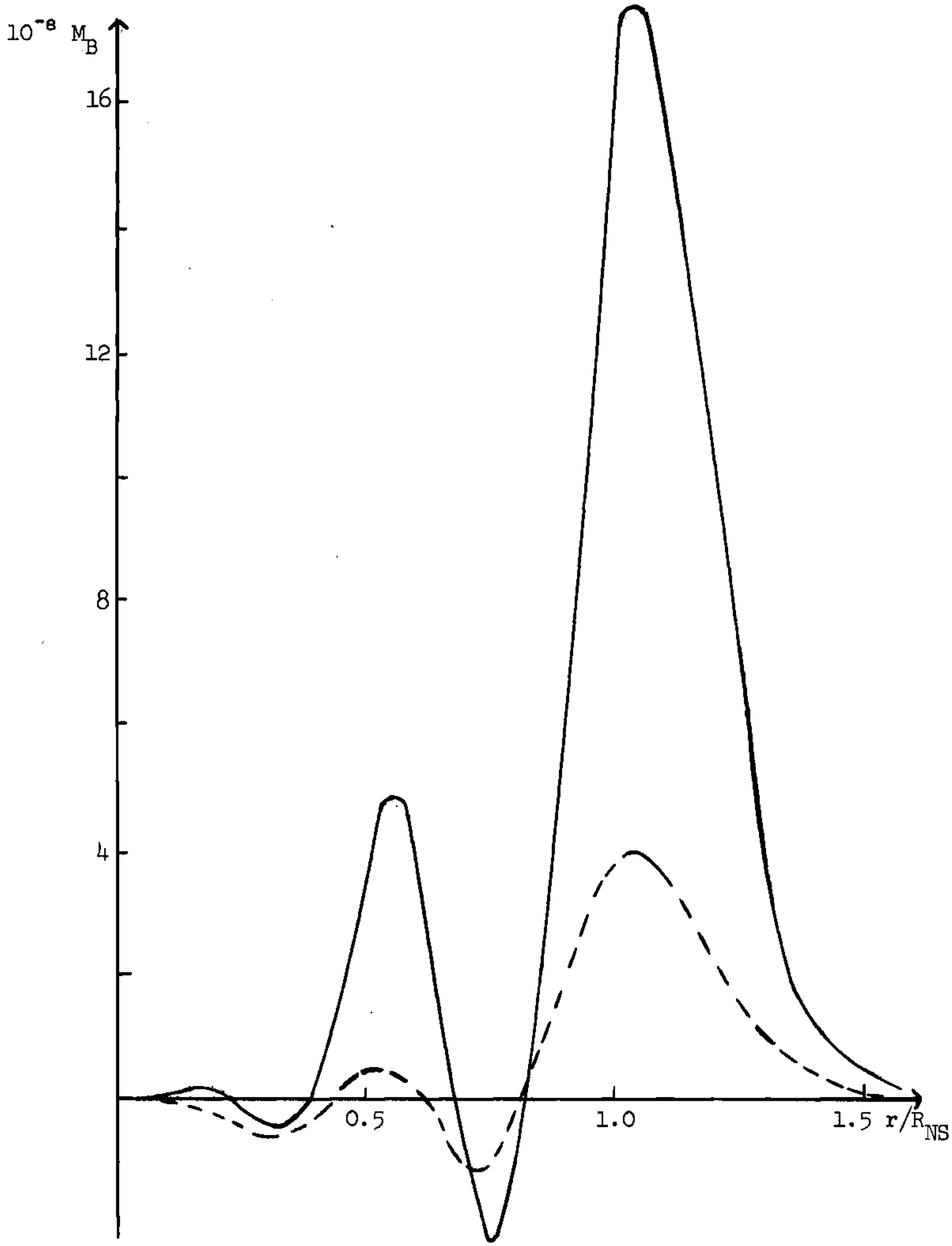


Figure 7. Concluded

$\|5\frac{1}{2}\ 63\rangle$		$\|4\frac{1}{2}\ 43\rangle$	
$\mu =$	0.45	$\mu =$	0.55
$\eta =$	6	$\eta =$	6
$a_{50} =$	0.1923	$a_{40} =$	0.163
$a_{30} =$	0.0842	$a_{20} =$	- 0.062
$a_{10} =$	- 0.4156	$a_{00} =$	- 0.279
$a_{51} =$	- 0.5260	$a_{41} =$	- 0.445
$a_{31} =$	0.6371	$a_{21} =$	0.833
$a_{11} =$	0.3171		

The solid curves are for the wave functions used by Bogdan,[22] who employed the Wood Saxton deformed potential wave functions of Faessler and Sheline; the Nilsson a's are obtained from the Faessler and Sheline Cj's by the expression given on page 57. They are for A = 185.

$\frac{1}{2}\,[501]_n$		$-\frac{1}{2}\,[41\text{-}1]_p$	
$\beta =$	0.3	$\beta =$	0.3
$a_{50} =$	- 0.0525	$a_{40} =$	- 0.3064
$a_{30} =$	- 0.2826	$a_{20} =$	0.6962
$a_{10} =$	- 0.2702	$a_{00} =$	- 0.5091
$a_{51} =$	- 0.06669	$a_{41} =$	0.3232
$a_{31} =$	- 0.3067	$a_{21} =$	- 0.2401
$a_{11} =$	0.8637		

In Bogdan's paper, he says he is using the $\frac{1}{2}\,[521]$ neutron state, but his coefficients are for the $\frac{1}{2}$ [501] state.

These radial nuclear matrix elements effectively go to zero at about one and a half nuclear surfaces. If

$$\frac{\int M \, dr}{\frac{3}{2} r_{NS}}$$

is small compared to the scale, this indicates cancellation and implies, as discussed in Chapter II, that the radial approximation would be bad.

For the dashed curve, there is cancellation for M_R and M_A, while for the solid curve there is no cancellation. Even when there is cancellation, we do not expect the calculations for the beta decay observables to be bad using the radial approximation, except possibly the shape, since the terms M_R, M_A, and M_K always appear summed together as indicated in the expression for M_{JL} in Chapter I. For this case, the term containing M_K is about 100 times the M_A term and about 20 times the M_R term. Hence the M_K term is dominant and it does not cancel.

To evaluate the beta decay observables using the radial approximation, the integrals which are listed in Table 9 are used. The symbols used are those which were defined previously, e.g.

$$\rho = a \, r$$

$$a = \left(\frac{M\omega_0 \ (\delta)}{\hbar}\right)^{\frac{1}{2}}$$

$$\omega_0 \ (\delta) = \overset{\circ}{\omega}_0 \left[1 - \frac{4}{3} \delta^2 - \frac{16}{27} \delta^3\right]^{-1/6}$$

$$\overset{\circ}{\omega}_0 = \frac{41}{.511} A^{-\frac{1}{3}} \text{ ; in RRE units}$$

$$B = \left(\frac{a_2}{a_1}\right)^2$$

$$A = \frac{1 + B}{2}$$

If $a_1 = a_2$, Nilsson's expression (reference 3, equation 41) for the radial integrals can be used.

Table 9. Some Radial Integrals

$$a_1^4 \int_0^\infty R_{44}(\rho_2) \quad R_{55}(\rho_1)\ r^3 dr = \frac{B^2}{A^{13/2}} \left(\frac{11}{2}\right)^{\frac{1}{2}}$$

$$a_1^2 \int_0^\infty R_{44}(\rho_2)\ D_+\ R_{55}(\rho_1)\ r^2 dr = \frac{B^2}{A^{13/2}} \left(\frac{11}{2}\right)^{\frac{1}{2}} (2A - 1)$$

If $a_1 = a_2$, then both the above integrals $= \left(\frac{11}{2}\right)^{\frac{1}{2}}$.

$$a_1^4 \int R_{44}(\rho_2) \quad R_{53}(\rho_1)\ r^3 dr = \frac{B^2}{2} \left(\frac{11}{A^{13/2}} - \frac{9}{A^{11/2}}\right)$$

$$a_1^2 \int R_{44}(\rho_2)\ D_-\ R_{53}(\rho_1)\ r^2 dr = \frac{B^2}{2} \left(\frac{13}{A^{11/2}} - \frac{11}{A^{13/2}}\right)$$

If $a_1 = a_2$, both = 1.

$$a_1^4 \int R_{42}(\rho_2) \quad R_{53}(\rho_1)\ r^3 dr = \frac{3\sqrt{2}}{4} \left[\frac{7B}{A^{9/2}} - \frac{9B^2 + 7B}{A^{11/2}} + \frac{11B^2}{A^{13/2}}\right]$$

Table 9. Some Radial Integrals (Continued)

$$a_1^2 \int R_{42}(\rho_2)\, D_+\, R_{53}(\rho_1)\, r^2 dr = \frac{3\sqrt{2}}{4}\left[\frac{14B}{A^{9/2}} - \frac{21B + 14B^2}{A^{9/2}} + \frac{7B + 27B^2}{A^{11/2}} - \frac{11B^2}{A^{13/2}}\right]$$

If $a_1 = a_2$, both $= \frac{3}{\sqrt{2}}$.

$$a_1^4 \int R_{42}(\rho_2) \quad R_{51}(\rho_1)\, r^3 dr = \frac{7}{4\sqrt{2}}\left[\frac{99}{7}\ \frac{B^2}{A^{13/2}} - \frac{9(2B^2 + B)}{A^{11/2}} + \frac{5B^2 + 14B}{A^{9/2}} - \frac{5B}{A^{7/2}}\right]$$

$$a_1^2 \int R_{42}(\rho_2)\, D_-\, R_{51}(\rho_1)\, r^2 dr = \frac{7}{4\sqrt{2}}\left[\frac{13B}{A^{7/2}} - \frac{22B + 13B^2}{A^{13/2}} + \frac{9}{A^{11/2}}\left(B + \frac{22}{7}B^2\right) - \frac{99}{7}\ \frac{B^2}{A^{13/2}}\right]$$

If $a_1 = a_2$, both $= \sqrt{2}$.

$$a_1^4 \int R_{40}(\rho_2) \quad R_{51}(\rho_1)\, r^2 dr = \frac{5\sqrt{14}}{8}\left[\frac{3}{2}\ \frac{1}{A^{5/2}} - \frac{3 + 5B}{A^{7/2}} + \frac{3 + 28B + 7B^2}{2A^{9/2}} - \frac{9}{5}\ \frac{5B + 7B^2}{A^{11/2}} + \frac{99}{10}\ \frac{B^2}{A^{13/2}}\right]$$

$$a_1^2 \int R_{40}(\rho_2)\, D_+\, R_{51}(\rho_1)\, r^2 dr = \frac{5\sqrt{14}}{8}\left[\frac{3}{A^{3/2}} - \frac{3}{2}\ \frac{5 + 4B}{A^{5/2}}\right.$$

Table 9. Some Radial Integrals (Concluded)

$$+ \frac{6 + 25B + 3B^2}{A^{7/2}} - \frac{3 + 56B + 35B^2}{2A^{9/2}}$$

$$+ \frac{9(B + \frac{14}{5} B^2)}{A^{11/2}} - \frac{99}{10} \frac{B^2}{A^{13/2}}\Big]$$

If $a_1 = a_2$, both $= \left(\frac{7}{2}\right)^{\frac{1}{2}}$.

Kotani Parameters

When the radial approximation is used, the expressions for the beta decay observables are often written in the form which uses the Kotani parameters. They are defined in Table 10 in terms of some of the commonly used symbols for the nuclear matrix elements.

For the case $a_1 = a_2$, the Kotani Λ parameter can be reduced to a compact form by using Table 9 and Table 7.

$$\Lambda = \frac{2R_{NS}}{\alpha Z}\,\frac{\sqrt{3}}{M}\,\frac{\int_0^\infty \langle V_1{}^{0}\cdot p\rangle\, r^2 dr}{\int_0^\infty \langle Y_1\rangle\, r^3 dr}$$

$$= \frac{2R_{NS}}{\alpha z}\,\frac{1}{M}\,a_1{}^2$$

$$= \frac{2}{\alpha Z}\,.4285\;\alpha\, A^{\frac{1}{3}}\,\frac{1}{M}\,\frac{M\omega_0(\delta)}{\hbar}$$

$$= \frac{2(.4285)}{\alpha Z}\,A^{\frac{1}{3}}\,\frac{41}{.511\,A^{\frac{1}{3}}}\,\frac{1}{\left[1 - \frac{4}{3}\,\delta^2 - \frac{16}{27}\,\delta^3\right]^{1/6}}$$

$$= \frac{68.8}{Z}\,\frac{1}{\left[1 - \frac{4}{3}\,\delta^2 - \frac{16}{27}\,\delta^3\right]^{1/6}}$$

which agrees with Tuong's[1] results where there it has been assumed $\delta \approx 0$, and where here we have the transition $N_i = 5 \rightarrow N_f = 4$.

Up until now, the δ we have used is the δ of the Nilsson paper. It is related to the ϵ of the Nilsson paper by

Table 10. The Definition of the Kotani Parameters

$$X = -\frac{C_V}{C_A}\frac{\int r}{\int B_{ij}} = -\frac{C_V}{C_A}\frac{M(r)}{M(B_{ij})} = -\frac{C_V}{2C_A}\frac{\int \langle Y_1 \rangle \; r^3 dr}{\int \langle V_2{}^1 \cdot \sigma \rangle \; r^3 dr} = -\frac{C_V}{C_A}\frac{1}{\sqrt{6}}\frac{\int M_R \, dr}{\int M_B \, dr}$$

$$U = \frac{\int i \; \sigma \times r}{\int B_{ij}} = \frac{i \; M(\sigma \times r)}{M(B_{ij})} = \frac{1}{\sqrt{2}}\frac{\int \langle V_1{}^1 \cdot \sigma \rangle \; r^3 dr}{\int \langle V_2{}^1 \cdot \sigma \rangle \; r^3 dr} = \frac{1}{\sqrt{6}}\frac{\int M_K \, dr}{\int M_B \, dr}$$

$$\Lambda = \frac{2R_{NS}}{\alpha Z}\frac{\int i\alpha}{\int r} = -\frac{i \; 2R_{NS}}{\alpha Z}\frac{M(\alpha)}{M(r)} = \frac{2R_{NS}}{\alpha Z}\frac{\sqrt{3}}{M}\frac{\int \langle V_1{}^0 \cdot p \rangle \; r^3 dr}{\int \langle Y_1 \rangle \; r^3 dr} = \frac{2R_{NS}}{\alpha Z}\frac{1}{\sqrt{2}}\frac{\int M_A \, dr}{\int M_R \, dr}$$

$$\omega_0(\delta) = \frac{\overset{\circ}{\omega}_0}{\left[1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right]^{1/6}} \equiv \omega_0(\epsilon) = \overset{\circ}{\omega}_0 \left[1 + \frac{1}{9}\epsilon^2\right]$$

Unfortunately, the ϵ of the Nilsson paper is the δ of the Mottelson-Nilsson paper.[3]

To see the sensitivity of the Kotani parameters, we calculate them for various Nilsson parameters for the decay of $_{69}Tm^{170}$. Note on Figures 3 and 4 of Mottelson and Nilsson[3] that, even when the η of the proton is equal to the η of the neutron, the δ of the proton is not equal to the δ of the neutron.

Here, as suggested by Gallagher and Solovier,[20] we use the same Nilsson states as Tuong.[1] That is, the odd proton is in the state $\frac{1}{2}+[411\downarrow] = |4\frac{1}{2}\ \#43>$, and the odd neutron is in the state $\frac{1}{2}-[521\downarrow] = |5\frac{1}{2}\ \#63>$. In the final state, both protons are in the $\frac{1}{2}+[411\downarrow]$ state. The Kotani parameters are also calculated using the Faessler and Sheline wave functions as suggested by Bogdan.[22] The odd proton is in the state $-\frac{1}{2}[41\text{-}1]$ and the odd neutron is in the state $\frac{1}{2}[521]$. In the final state, both protons are in the $-\frac{1}{2}[41\text{-}1]$ state. When Bogdan wrote the expansion coefficients for the $\frac{1}{2}[521]$ state, he actually used the coefficients for the $\frac{1}{2}[501]$ state. The Kotani parameters were calculated for this transition as well. Actually on the energy level plot, the $\frac{1}{2}[501]$ state should represent the 125^{th} neutron, not the 101^{st} neutron.

Table 11. The Kotani Parameters for Various Nuclear Parameters for the Decay of Tm^{170}

$\frac{1}{2}$ - [521↓] → $\frac{1}{2}$ + [411↓]					X	U	Λ
η_N	μ_N	η_p	μ_p	B			
4	0.45	4	0.55	1	-0.098	4.23	0.986
				1.003	-0.102	4.19	1.076
6	0.45	4	0.55	1	-0.124	1.73	0.992
				.997	-0.113	1.79	1.270
4	0.45	6	0.55	1	0.060	2.79	0.986
				1.011	0.052	2.87	1.292
6	0.45	6	0.55	1	-0.036	1.52	0.992
					-0.039	1.52	0.921
6	0.70	6	0.55	1	-0.359	2.13	0.992
				1.005	-0.366	2.15	0.996
$\frac{1}{2}$ $[521]_n$ → - $\frac{1}{2}$ $[41\text{-}1]_p$ β = 0.3				1	0.417	-0.436	0.992
$\frac{1}{2}$ $[501]_n$ → - $\frac{1}{2}$ $[41\text{-}1]_p$ β = 0.3				1	-0.267	0.265	0.992

CHAPTER IV

THE CALCULATED VALUES OF BETA DECAY OBSERVABLES FOR Re^{186} AND Tm^{170}

In this chapter the values obtained from experiment will be compared with those obtained using the two particle intrinsic wave functions of Chapter III for the first forbidden beta decay of Re^{186} and Tm^{170}, which both have the same decay scheme as given in Figure 1.

The observables are calculated using the terms kept by Morita and Morita, first using the radial approximation, i.e.

$$\int_0^\infty A_i(JLK\kappa\nu) <O_J^L> r^2 dr \simeq \left[\frac{A_i(JLK\kappa\nu)}{r^L}\right]_{R_{NS}} \int_0^\infty <O_J^L> r^{2+L} dr$$

then using what will be called the Buhring approximation

$$\int_0^\infty A_i(JLK\kappa\nu) <O_J^L> r^2 dr \simeq \left[\frac{A_i(JLK\kappa\nu)}{r^L}\right]_{R_{NS}}$$

$$\frac{1}{2\left(1 + \frac{{}^1\alpha_i(JLK\kappa\nu)}{{}^0\alpha_i(JLK\kappa\nu)}\right)} \left\{\left(2 + \frac{{}^1\alpha_i(JLK\kappa\nu)}{{}^0\alpha_i(JLK\kappa\nu)}\right) \int_0^\infty <O_J^L> r^{2+L} dr\right.$$

$$\left. + \frac{{}^1\alpha_i(JLK\kappa\nu)}{{}^0\alpha_i(JLK\kappa\nu)} \int_0^\infty <O_J^L> \frac{r^{4+L}}{R_{NS}^2} dr \right\}$$

and finally assuming the electron sees a uniform charge distribution and numerically performing these integrals as discussed in Chapter II.

Since the calculated beta-gamma A_2 coefficient for Tm^{170} does not agree with the experimental value, these observables are calculated for various initial and final nuclear shapes as well as for two cases of the Faessler and Sheline wave functions which were used by Bogdan.

The wave functions used are the same as Tuong's and are listed in Table 12.

The first observable one can calculate using these wave functions is the difference in energy between the initial and final energy levels of the transforming nucleon.

For example, from the energy level diagram in the Nilsson paper[3] for the parameter $\eta = 6$, corresponding to the decay of Tm^{170}, the energy of the $\frac{1}{2}$ - #63 state is 6.1 $\hbar\omega_0(\delta)$, and the $\frac{1}{2}$ + #43 state is 5.45 $\hbar\omega_0(\delta)$, with $\delta = .29$. Where $\hbar\omega_0(\delta)$ is defined as $\hbar\omega_0 \left(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right)^{1/6}$ and $\hbar\omega_0 \sim 41/A^{\frac{1}{3}}$ MeV. Hence for A = 170

$$E_i - E_f = .65 \frac{41}{(170)^{1/3}} \frac{1}{(1 - .127)^{1/6}} = 4.9 \text{ MeV}$$

This does not agree with the energy liberated experimentally

$$\Delta E = m_e c^2 + T \max = (.511 + .833) \text{ MeV} = 1.344 \text{ MeV}$$

When using the adjusted energy level diagram for protons in the Mottelson-Nilsson paper again for $\eta = 6$, the energy of the $\frac{1}{2}$ - #63 state is again 6.1 $\hbar\omega_0(\delta)$ where $\delta = .29$ and the $\frac{1}{2}$ + 411 state is 5.45 $\hbar\omega_0(\delta)$ where

Table 12. The Wave Functions and End Point Energies Used for the Decay of Tm^{170} and Re^{186}

For ${}_{69}Tm^{170}$	Mottelson & Nilsson, Faessler & Sheline Notation	Nilsson Notation
101 neutron	$\frac{1}{2}$ - [521↓]	$\frac{1}{2}$ - 63
69 and 70 proton	$\frac{1}{2}$ + [411↓]	$\frac{1}{2}$ + 43
For ${}_{75}Re^{186}$		
111 neutron	$\frac{3}{2}$ - [512↓]	$\frac{3}{2}$ - 62
75 and 76 proton	$\frac{5}{2}$ + [402↓]	$\frac{5}{2}$ + 31

From Dulaney, et al.[24]

	T_{max} in keV i → I	T_{max} i → f	E I → f	Half life of intermediate state 10^{-9} secs	% i → I	% i → f
Tm^{170}	883	967	84.2	1.6	22	78
Re^{186}	934	1071	137	0.5	23.1	73.0

δ = .36 remembering that the δ in the Mottelson-Nilsson paper is the ϵ of the Nilsson paper, where ϵ is related to the δ of the Nilsson paper by

$$\delta_{MN} = \delta_N + \frac{1}{6}\,\delta_N^2$$

and

$$\overset{\circ}{\omega}_0(\epsilon) = \omega_0\left(1 + \frac{1}{9}\,\epsilon^2\right)$$

$$E_i - E_f = 4.6 \text{ MeV}$$

Doing the same thing for Re^{186}, one gets similar results.

Comparison of the observable for the three different ways of evaluation of the radial integral is given in Table 13 for the case of the decay of Tm^{170}, where

$$\frac{\ln 2}{t_{\frac{1}{2}}} = \lambda = (2\pi)^3\;(C_V g)^2 \int_1^{W_0} a^3\,|H'|^2\;\frac{W\,q^2}{p}\,dW$$

and where the normalized shape factor is normalized to p ≡ 1. We see that at p = 1 the radial approximation gives an absolute shape $|H'|^2$ that is 10 percent too large, which means that, if $|H'|^2$ is consistently 10 percent larger for all momenta when using the radial approximation, then the half life calculated will be 10 percent smaller using the radial approximation. Similarly, using the Buhring approximation, one would calculate a half life that would be four percent smaller than the value obtained by numerically evaluating the radial integral.

We see that using either approximation, the error introduced in the normalized shape factor is about two percent.

Table 13. Comparison of Calculated Observables for Tm^{170}

$\frac{1}{2} - [521] \rightarrow \frac{1}{2} + [411]$

$\eta = 6$	$\eta = 6$
$\mu = 0.45$	$\mu = 0.55$
$\delta = 0.29$	$\delta = 0.29$

	Morita and Morita Approximation	Two Term Buhring Approximation	Numerical Calculation
$10^{15}\lvert H'\rvert^2$	3.49	3.29	3.17
$\frac{N_S\ (p = 2.4)}{N_S\ (p = 1)}$	1.12	1.12	1.14
$A_2\ (p = 1)$	-0.0134	-0.0138	-0.0145
$A_2\ (p = 2.4)$	-0.0403	-0.0414	-0.0424

In calculating the A_2 coefficient, we see the error introduced using the radial approximation is about six percent, whereas the Buhring approximation introduces only about three percent.

These errors, for this particular case, are typical of what was encountered for the other cases calculated in this paper.

Figure 8 gives a comparison of the values of the A_2 coefficient measured by Dulaney, et al.[24] versus those calculated using the above wave functions obtained from the Mottelson-Nilsson paper both with and without the radial approximation.

The A_2 coefficient was also evaluated for various Nilsson parameters to test its sensitivity and the results are indicated in Table 14. It was also evaluated using the Faessler and Sheline wave functions that were used by Bogdan. Actually, Bogdan used the $\frac{1}{2}$ $[501]_n$ wave function for the 101^{st} neutron where, according to Faessler and Scheline, this should describe the 125^{th} neutron.

Notice that at high momentum (p = 2.4) the A_2 coefficient changes by a factor of two depending on the initial and final Nilsson parameters.

When looking at the Faessler and Sheline results, it must be remembered that they tabulated their wave functions for A = 185 not for 170 nucleons that are in ${}^{101}_{69}Tm^{170}$. Using the [521] wave function which should represent the 101^{st} neutron, it even gives the wrong sign on the A_2 coefficient, whereas using the [501] wave function, which represents the 125^{th} neutron and lies a couple of MeV higher, we get about the experimentally observed value.

The comparison of the normalized shape factor with experiment is

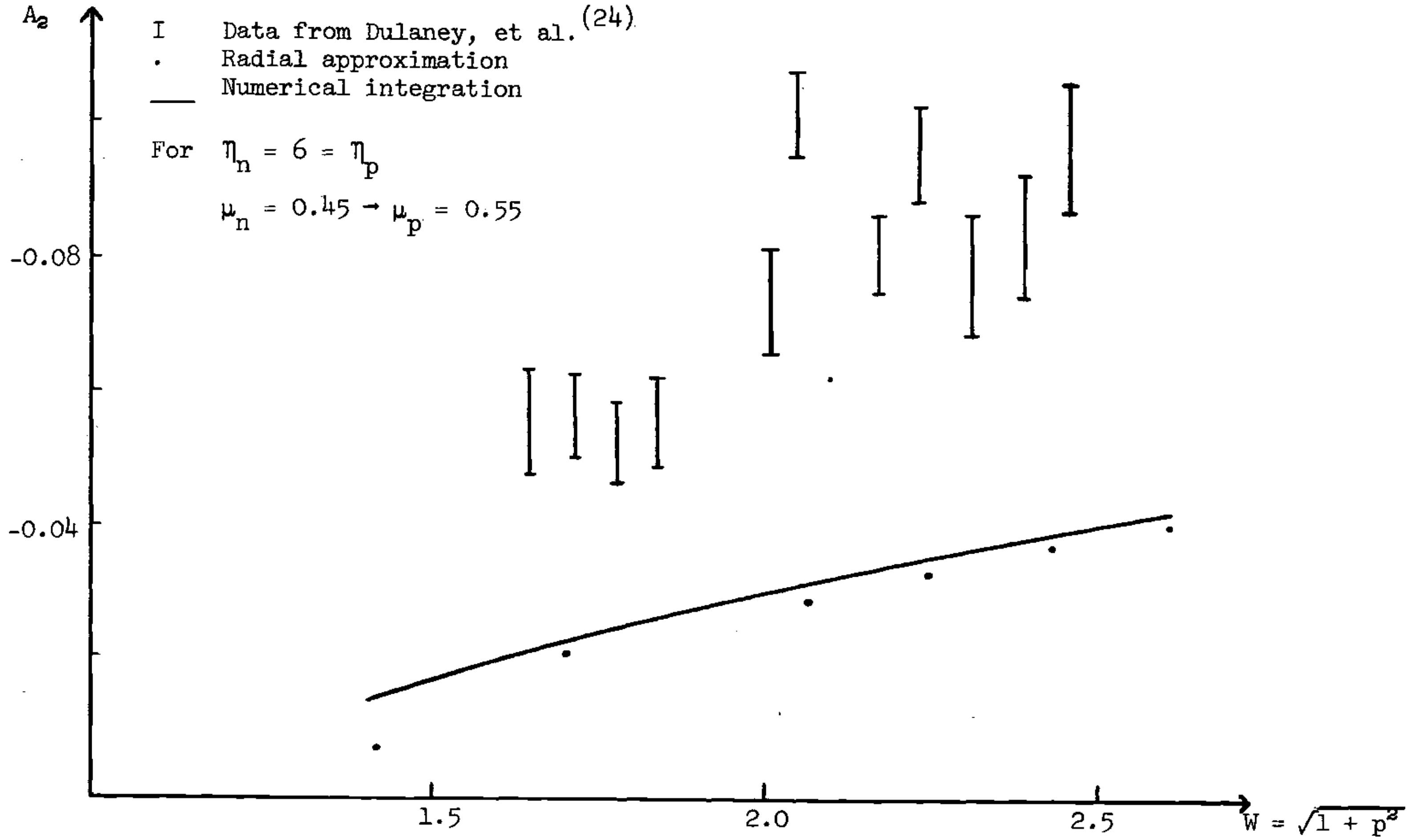


Figure 8. A_2 Coefficient for Tm^{170}

Table 14. The A_2 Coefficient for Tm^{170}

				p = 1.0	p = 2.4
For the Mottelson-Nillson Wave Functions					
$\frac{1}{2} - [521] \rightarrow \frac{1}{2} + [411]$					
η_N	μ_N	η_p	μ_p		
6	0.70	6	0.55	- 0.0145	- 0.0426
6	0.45	6	0.55	- 0.0146	- 0.0426
4	0.45	6	0.55	- 0.0111	- 0.0225
6	0.45	4	0.55	- 0.0141	- 0.0414
4	0.45	4	0.55	- 0.0104	- 0.0305
For the Faessler and Sheline Wave Functions					
$\frac{1}{2}[521]_n \rightarrow -\frac{1}{2}[41\text{-}1]_p$				+ 0.0012	+ 0.0068
$\frac{1}{2}[501]_n \rightarrow -\frac{1}{2}[41\text{-}1]_p$				- 0.0457	- 0.135

just as bad as that of the A_2 coefficient for this decay. Experimentally it is observed that this has an allowed shape, i.e., the normalized shape factor is independent of energy ($N_s(p) = 1$). Calculating the normalized shape factor for all the above initial and final Nilsson parameters, one obtains

$$\frac{N_s(2.4)}{N_s(p = 1)} = 1.13 \pm 1\%$$

However, using the Nilsson wave functions, we get much better agreement for the decay Re^{188}, as shown in Figures 9 and 10 for $\eta_n = 6 = \eta_p$ and $\mu_n = .45 = \mu_p$.

Again the error introduced using the radial approximation is small ($\sim$ one percent) for the normalized shape factor. Again the absolute shape, $|H'|^2$ is much more sensitive to the radial approximation, but this time it is 15 percent smaller, which means that, if $|H'|^2$ is 15 percent smaller over all momenta, the half life calculated using the radial approximation will be 15 percent too large.

Using the radial approximation to calculate the A_2 coefficient for this case, we get a result that is five percent too large. The calculated A_2 coefficient probably does not fit the experimental results as well as indicated in Figure 10, because the expressions used for the calculation of the A_2 coefficient assumed that the intermediate state decayed immediately where experimentally it takes a half of a nanosecond. So if anything, one would expect that the calculated A_2 coefficient would be greater than that observed since the intermediate state will have some time to randomize itself.

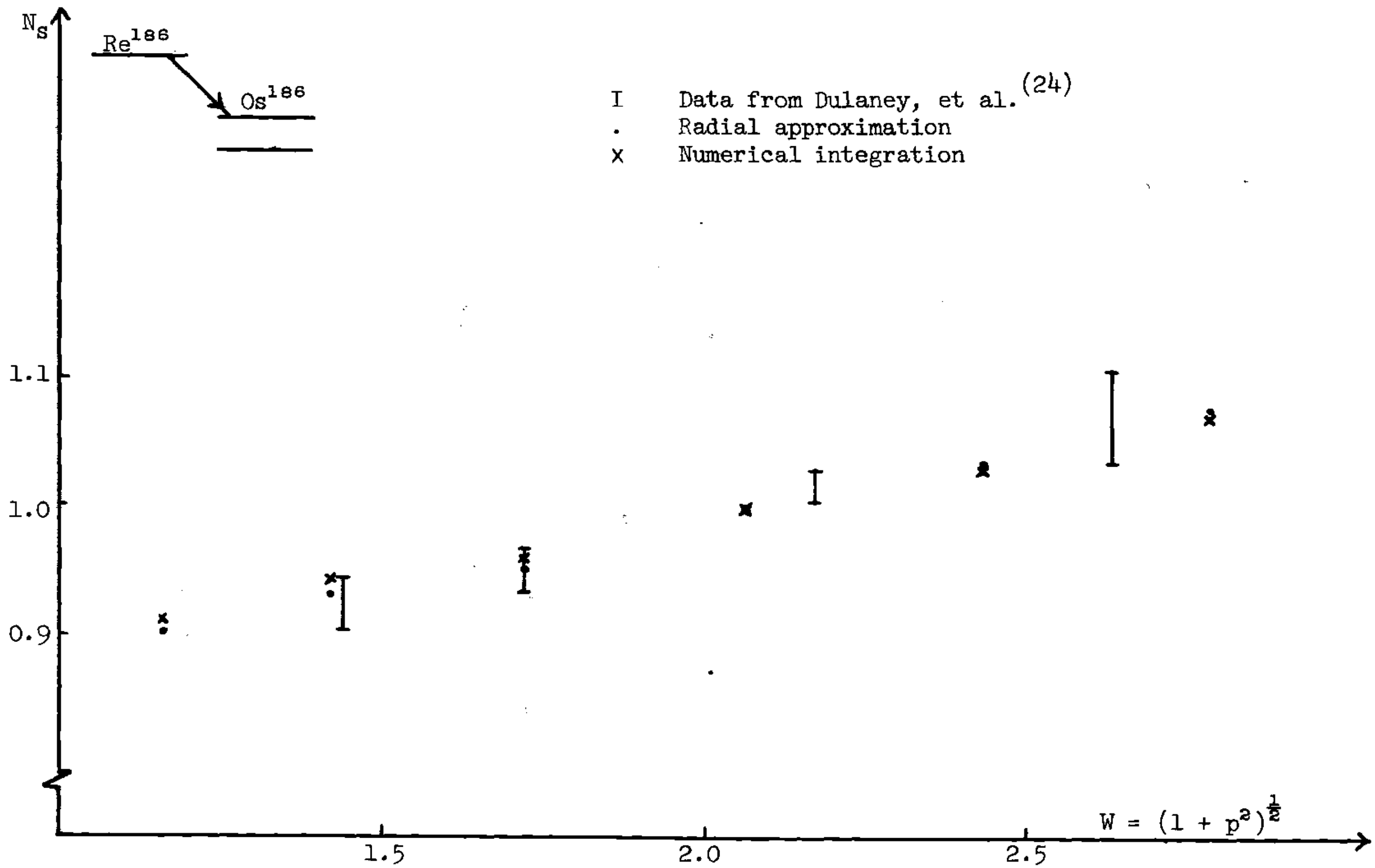


Figure 9. The Normalized Shape for Re^{186}

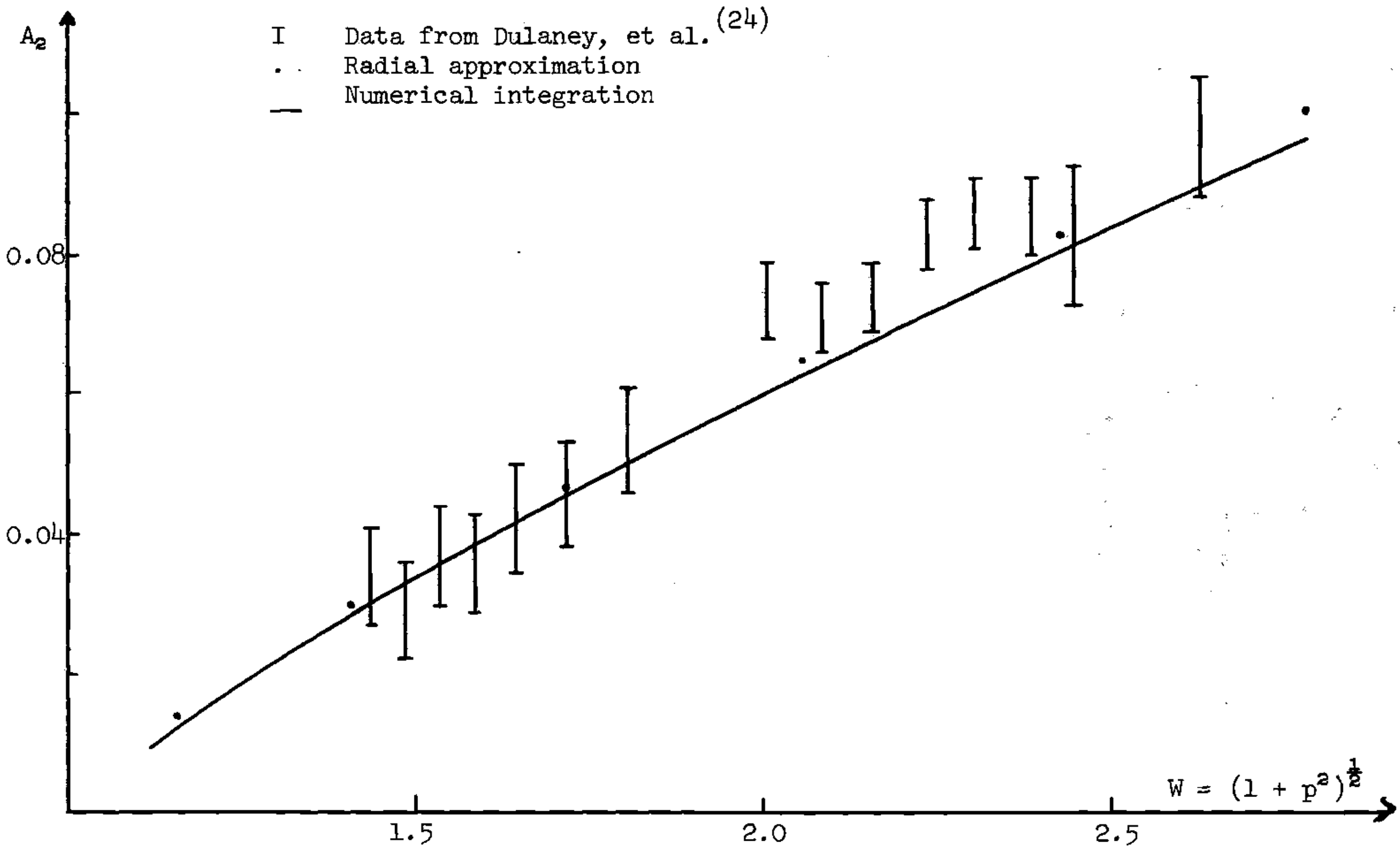


Figure 10. The A_2 Coefficient for Re^{186}

On Figure 11 is a plot of the normalized shape factor for the beta decay of Re^{186} to the ground state of Os^{186}. Again the error introduced by using the radial approximation is small and again the absolute shape is more sensitive to the radial approximation. For this case, it is nine percent low, indicating that the half life calculated using the radial approximation would be nine percent too large. Hence, the ratio of half lives of the decay to the first excited state of Os^{186} relative to the decay to the ground state would be six percent too large if the radial approximation is used.

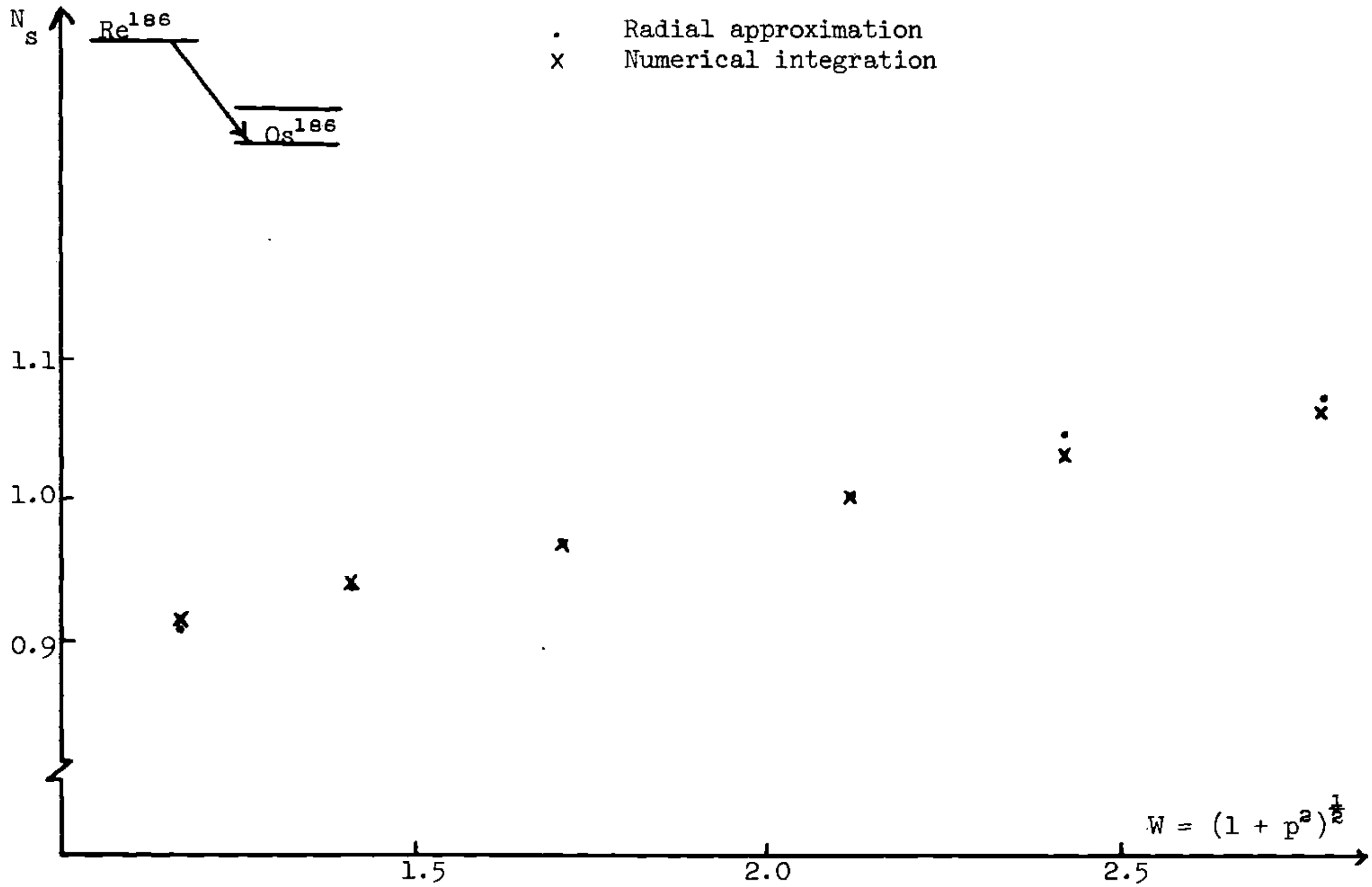


Figure 11. The Normalized Shape for the Decay of Re^{186} to the Ground State of Os^{186}

CHAPTER V

COMMENTS AND CONCLUSIONS

As seen in Chapter IV, the observables calculated, for similar beta decays of Re^{188} and Tm^{170}. are relatively insensitive to the radial approximation for the nuclear wave functions used. The error introduced using the radial approximation to calculate the A_2 coefficient and the normalized shape factor are within experimental error. The calculation most sensitive to the approximation is the absolute shape which can be used to determine the half life. However, the ratio of the half life of the decay to the first excited state to the half life of the decay to the ground state is much less sensitive. Hence the bad fit for the A_2 coefficient of Tm^{170} is not due to the radial approximations but to the nuclear wave function used.

This insensitivity is due to the largest terms in the calculation being insensitive to the approximation. For instance, the largest term appearing in the calculation for the shape is the term

$$\Big| \sum_L M_{1L} \; (\kappa = -1, \; \kappa_\nu = 1) \Big|^2$$

which in the radial approximation is proportional to

$$\Big| - \left[\frac{j_0 \; G_{-1} \; - \; j_1 \; G_{-1}}{r} \right]_{NS} \int \vec{r} + \frac{C_A}{C_V} \left[\frac{j_0 \; F_{-1} \; j_1 \; G_{-1}}{r} \right]_{NS} \int i \; \vec{\sigma} x \vec{r}$$

$$- j_0 \, G_{-1} \int i \, \vec{\alpha}|^2$$

For the decay of Tm^{170} with p = 1 the percent error introduced by the radial approximation into the magnitude of these terms is tabulated below.

For the	$\vec{r}$	term	-49%
	$i \, \vec{\sigma} x \vec{r}$	term	5%
	$\vec{\alpha}$	term	53%

Using the Kotani parameters given in Table 11, for this case we have the values

$$\int \vec{r} \simeq \frac{2R_{NS}}{\alpha Z} \int i \, \vec{\alpha}$$

$$\int i \, \sigma x r \simeq 20 \int \vec{r}$$

Since the magnitude of the $\int i \, \sigma x r$ term is about 20 times the size of the other two, it dominates the

$$| \sum_{L} M_{1L} \, (-1,1) |^2$$

term appearing in the shape calculation.

Similar results are obtained when one looks at the largest term in the A_2 coefficient, i.e.

$$\sum_{L} M_{1L} \, (11) \sum_{L'} M_{1L'}{}^{*} \, (-21)$$

The error introduced into the $\sum M_{1L}$ (11) term is just about the same as the $\sum M_{1L}$ (-11) term. The M_{1L} (-21) term in the radial approximation is proportional to

$$\left[\frac{j_0 \; G_{-2}}{r}\right]_{NS} \int \vec{r} + \frac{C_A}{2C_V}\left[\frac{j_0 \; G_{-2}}{r}\right]_{NS} \int i \; \vec{\sigma} x \vec{r}$$

For the same conditions as above, the error introduced by the radial approximation is

for the $\int \vec{r}$ term -14%

and for the $\int i \; \vec{\sigma} x \vec{r}$ term 2%

These results are consistent with the arguments of Chapter II and Figure 7. On those graphs, we see that M_r and M_A suffer cancellation while M_K does not. The reason that there is less error in using the radial approximation for the $\int \vec{r}$ term in the M_{1L} (-21) term than in the M_{1L} (-11) term is that the slope of $\frac{G_{-2}}{r}$ is smaller than $\frac{F_{-1}}{r}$ in the region zero to $2R_{NS}$.

To get an idea of the sensitivity of individual terms to the approximations, let us look at the term

$$I(0) = \int F_{-1} \langle O_J{}^L \rangle \, r^2 dr$$

Using the radial approximation, this is about equal to

$$I_R(0) \equiv \left[\frac{F_{-1}}{r^L}\right]_{NS} \int \langle O_J{}^L \rangle \, r^{2+L} dr$$

or the Buhring approximation

$$I_B(0) = \left[\frac{F_{-1}}{r^L}\right]_{NS} \int \frac{2 + \frac{a_1}{a_0} + \frac{a_1}{a_0}\left(\frac{r}{r_{NS}}\right)^2}{2\left(1 + \frac{a_1}{a_0}\right)} \langle O_J{}^L \rangle \, r^{2+L} dr$$

For the above case the results are

$$\begin{aligned}
I_R(\vec{r}) &= 0.51\ I\ (\vec{r})\\
I_B(\vec{r}) &= 0.82\ I\ (\vec{r})\\
I_R(i\ \vec{\sigma}x\vec{r}) &= 1.05\ I\ (i\ \vec{\sigma}x\vec{r})\\
I_B(i\ \vec{\sigma}x\vec{r}) &= 1.04\ I\ (i\ \vec{\sigma}x\vec{r})\\
I_R(\vec{\alpha}) &= 1.53\ I\ (\vec{\alpha})\\
I_B(\vec{\alpha}) &= 0.75\ I\ (\vec{\alpha})
\end{aligned}$$

Now apparently the usual two term Buhring approximation that is used is for this particular case

$$I_B'(0) = \left[\frac{F_{-1}}{r^L}\right]_{NS} \int \frac{1 + \frac{a_1}{a_0}\left(\frac{r}{r_{NS}}\right)^2}{1 + \frac{a_1}{a_0}} \langle O_J{}^L \rangle \, r^{2+L} dr$$

which for the above case yields

$$\begin{aligned}
I_B'(r) &= 1.15\ I\ (r)\\
I_B'(i\ \sigma xr) &= 1.01\ I\ (i\ \sigma xr)\\
I_B'(\alpha) &= 1.37\ I\ (\alpha)
\end{aligned}$$

This is surprizing because of its sensitivity and, since in Chapter II, we saw that the average between treating the lepton part as a constant and using the two term Buhring expansion better fits the value obtained by numerical techniques.

In conclusion, we see that the discrepancy between the experimentally observed A_2 coefficient and that calculated using the two particle Nilsson wave functions for the decay of Tm^{170} is not due to the radial approximation.

The above analysis suggests two methods to test the validity of the radial approximation for particular calculations. The first method is indicated in Chapter II where the nuclear contribution to the matrix element is plotted. If the dominant terms in the calculation suffered cancellation, then the radial approximation would be invalid.

The other and perhaps easier method would be to calculate the observable using the radial approximation and also using the two term Buhring approximation.

It appears that the "averaged" or the "usual" two term Buhring approximation works equally well, even though the averaged better fits the numerical values of the lepton values. If there is an appreciable difference between these two calculations, it would indicate that probably both approximations are in error. Then, not only would the radial integrals have to be evaluated numerically, but some of the smaller terms appearing in $\sum_L M_{JL}(\kappa\kappa_\nu)$, which were neglected in Chapter I, would probably now be significant.

APPENDIX I

In order to calculate any beta decay observables, the matrix element H_{Ii} must be evaluated.

Here an expression for this matrix element will be developed using the V-A law and assuming the nucleons can be described by non-relativistic wave functions. The method employed will give the same results to order 1/m that Rose and Osburn[9] obtained using the Foldy-Wouthuysen transformation.

The V-A beta-decay matrix element as given by Konopinski (reference 5, p. 120) is

$$H_{Ii} = \int \psi_I^+ \; H \; \psi_i \; d\tau = \int h d\tau$$

$$= \int \psi_I \;\; \sqrt{2}\, g \sum_{a=1}^{A} \{ (C_V - C_A \gamma_5^a) \; \beta^a \; \gamma_\alpha^a \; \tau_+^a \; [J_\alpha \; (e^\nu)]_{\vec{r}^a}$$

$$+ \text{h.c.} \} \; \psi_i \; d\tau$$

The relations for the Dirac operators in Konopinski's notation are as follows.

$$\gamma_\mu = (-i \; \beta \; \gamma_i, \; \beta)$$

$$\gamma_\mu \; \gamma_\nu + \gamma_\nu \; \gamma_\mu = 2 \; \delta_{\mu\gamma}$$

$$\gamma_5^2 = 1$$

$$\gamma_5 = \gamma_1 \ \gamma_2 \ \gamma_3 \ \gamma_4 = -\rho_1 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\beta = \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\alpha} = -\vec{\sigma} \ \gamma_5 = \rho_1 \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\vec{\sigma} = \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

The lepton current can be written in terms of the electron and neutrino wave functions as

$$J_\alpha \ (e^\nu) = \psi_e^{\ +} \ (\pm \ Z) \ \beta \ \gamma_\alpha \frac{1 \pm \gamma_5}{2} \ \psi_\nu^{\ c} \ ; \quad \text{for } e^\mp \text{ emission}$$

Suppressing the sum over the A nucleons and the isotopic spin operator $\tau_+^{\ a}$, and dropping the hermitian conjugate (h.c.), which describes positron emission, the Hamiltonian density can be rewritten as

$$h = \frac{g}{\sqrt{2}} \Big\{ \psi_I^{\ +} \ (C_V \vec{\alpha} + C_A \vec{\sigma}) \ \psi_i \cdot \ \psi_e^{\ +} \ \vec{\sigma} \ (\pm \ 1 + \gamma_5) \ \gamma_\nu^{\ c}$$

$$+ \ \psi_I^{\ +} \ (C_V - C_A \ \gamma_5) \ \psi_i \quad \psi_e^{\ +} \ (1 \pm \gamma_5) \ \gamma_\nu^{\ c} \Big\}$$

Since the nucleons, electrons, and neutrinos are spin one-half particles, their wave functions should satisfy the Dirac equation

$$W\psi = (c\alpha \cdot p + \beta mc^2 + V) \ \psi$$

Using $\psi = \binom{U}{V}$, the Dirac equation can be written as

$$W \begin{pmatrix} U \\ v \end{pmatrix} = \left\{ c \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot p + mc^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + V \right\} \begin{pmatrix} U \\ v \end{pmatrix}$$

Solving for v one gets

$$\psi = \begin{pmatrix} 1 \\ \dfrac{c\ \sigma \cdot\ p}{W-V + mc^2} \end{pmatrix} U$$

Following Konopinski (reference 5, p. 224), a good approximation for nucleons in a nucleus should be

$$|V| \ll mc^2 \quad \text{and} \quad |p| \ll mc$$

Hence the nucleon wave function can be approximated by

$$\psi \simeq \begin{pmatrix} 1 \\ \dfrac{\sigma \cdot p}{2mc} \end{pmatrix} U$$

Using this approximation and defining

$$\vec{A} = \psi_e^{+} (\pm Z)\ \vec{\sigma}\ (\pm 1 + \gamma_5)\ \psi_\nu^{c}$$

and

$$A_4 = \psi_e^{+} (\pm Z) \quad (1 \pm \gamma_5)\ \psi_\nu^{c}$$

then h can be written as

$$h = \frac{g}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ \dfrac{\sigma \cdot p}{2m_I c} \end{pmatrix} U_I \right]^{+} \left\{ (C_V \vec{\alpha} + C_A \vec{\sigma}) \cdot A \right.$$

$$\left. + (C_V - C_A\ \gamma_5)\ A_4 \right\} \begin{pmatrix} 1 \\ \dfrac{\sigma \cdot p}{2m_i c} \end{pmatrix} U_i$$

We now keep terms of up to order $1/m$ since $1/m^2$ terms give contributions of the same order as $1/m$ terms would in second order perturbation theory. Using $m_I \simeq m_i \equiv M$ and the following identities

$$\vec{\sigma}\cdot\vec{p}\ \vec{\sigma}\cdot\vec{q} = \vec{p}\cdot\vec{q} + i\ \vec{\sigma}\cdot(\vec{p} \times \vec{q})$$

$$\vec{p}\cdot(U_i\ \vec{A}) = U_i\ \vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\ U_i$$

$$\vec{p} \times (U_i\ \vec{A}) = U_i\ \vec{p} \times \vec{A} - \vec{A} \times \vec{p}\ U_i$$

the Hamiltonian density can be rewritten as

$$h = \frac{g}{\sqrt{2}} \Big\{ C_V\ (A\cdot U_I^{+}\ \frac{p}{Mc}\ U_i + U_I^{+}\ U_i\ \frac{p}{2Mc}\cdot A + i\ U_I^{+}\ \sigma\ U_i\cdot\frac{p}{2Mc} \times A)$$

$$+ C_A\ A\cdot U_I^{+}\ \sigma\ U_i + C_V\ A_4\ U_I^{+}\ U_i$$

$$+ C_A\ (A_4\ U_I^{+}\ \frac{\sigma\cdot p}{Mc}\ U_i + U_I^{+}\ \sigma\ U_i\cdot\frac{p}{2Mc}\ A_4)\Big\}$$

This Hamiltonian density with non-relativistic nuclear wave functions agrees with that of Rose and Osburn,[9] who used a Foldy-Wouthuysen transformation. If only the first term in the parenthesis is kept, the non-relativistic limit of the nuclear operators can be written as

$$\vec{\alpha} \xrightarrow[NR]{} \frac{\vec{p}}{Mc}$$

$$\vec{\sigma} \longrightarrow \vec{\sigma}$$

$$1 \longrightarrow 1$$

$$\gamma_5 \longrightarrow -\frac{\vec{\sigma}\cdot\vec{p}}{Mc}$$

APPENDIX II

In this section, a multipole expansion will be made for the lepton contribution, $A_\mu(A,A_4)$, to the beta decay matrix element, for the electron seeing a spherical potential. Angular momentum seems to be a good quantum number for nuclear states. Hence, nuclear states are presented in a spherical representation. Thus the natural representation for the leptons would also then be the angular momentum representation.

To expand the lepton contribution

$$\vec{A} = \psi_e^{+} (\pm Z) \vec{\sigma} (\pm 1 + \gamma_5) \psi_\nu^{c}$$

$$A_4 = \psi_e^{+} (\pm Z) \quad (1 \pm \gamma_5) \psi_\nu^{c}$$

we make a multipole expansion of the electron and neutrino wave functions. The neutrino is treated as a free particle and has a momentum $\vec{q}$ and spin σ_ν along $\vec{q}$. It is a plane-wave, which can be expanded in spherical waves to yield (reference 7, p. 1059)

$$\psi_\nu^{c} (q,\sigma) = \sum_{\kappa_\nu \mu_\nu} \sqrt{4\pi (2\ell_\nu + 1)} \; (\ell_\nu \tfrac{1}{2} j : 0 \; \sigma_\nu \; \sigma_\nu)$$

$$D^{j_\nu}_{\mu_\nu \sigma_\nu} (Z \rightarrow q) \; \psi^{c}_{\kappa_\nu \mu_\nu} (\vec{r})$$

$(j_1 \; j_2 \; j_3 : m_1 \; m_2 \; m_3)$ is the Clebsch-Gordan coefficient.

The total angular momentum quantum number is j, ℓ is the orbital angular momentum, and $\sigma = \frac{1}{2}$ the spin. $D^{j}_{m'm}$ is the rotation matrix. ψ^{c} is the charge conjugate wave function for anti-particles and is related to the Dirac wave function by[5, 13]

$$\psi^{c}_{\kappa\mu} = \rho_2 \; \Sigma_2 \; \psi^{*}_{\kappa\mu} = \gamma_2 \; \psi^{*}_{\kappa\mu}$$

$\psi_{\kappa\mu}$ is the solution of the Dirac equation for a central field

$$\psi_{\kappa\mu} = \begin{pmatrix} G & (|r|) & X_{\kappa\mu} \\ iF & (r) & X_{\kappa\mu} \end{pmatrix}$$

where F and G are real, and $\psi_{\kappa\mu}$ is an eigenstate of the Dirac angular momentum operator

$$K \; \psi_{\kappa\mu} = \beta \; (\vec{\sigma}\cdot\vec{L} + 1) \; \psi_{\kappa\mu} = K \; \psi_{\kappa\mu}$$

and

$$J_Z \; \psi_{\kappa\mu} = \mu \; \psi_{\kappa\mu}$$

κ has the following eigenvalues and relations.

$$\kappa = \pm 1, \pm 2 \; \text{---}$$

$$j = |\kappa| - \tfrac{1}{2}$$

$$\ell = \begin{cases} j + \frac{1}{2} = \kappa \; ; \; \kappa > 0 \\ j - \frac{1}{2} = -(\kappa + 1) \; ; \; \kappa < 0 \end{cases}$$

$$\bar{\ell} = \ell\,(-\kappa) = \ell - \frac{\kappa}{|\kappa|}$$

The "spinor spherical harmonic" (reference 5, p. 60) is defined as

$$X_{\kappa\mu}(\theta,\Phi) = \sum_{m=\pm\frac{1}{2}} (\ell\ \tfrac{1}{2}j: \mu-m,\ m,\ \mu)\ \chi_m\ Y_{\ell,\mu-m}$$

Where Y_{LM} is the spherical harmonic and the spinors are

$$\chi_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \chi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}$$

F and G satisfy the coupled pair of first order Dirac radial equations

$$\frac{dF_\kappa}{dr} = \frac{\kappa - 1}{r} F_\kappa - (W - 1 - V(r))\ G_\kappa(r)$$

$$\frac{dG_\kappa}{dr} = -\frac{\kappa + 1}{r} G_\kappa + (W + 1 - V(r))\ F_\kappa(r)$$

These are the same differential equations that Bhalla and Rose[14] have used.

For free massless particles

$$W \mp 1 - V(r) \rightarrow W = q$$

and the solutions of these equations are the spherical Bessel functions $j_\ell\ (qr)$.

Hence, for the free massless neutrino

$$\psi^{c}_{\kappa_\nu \mu_\nu} = \frac{q}{\sqrt{\pi}} (-)^{\mu_\nu + \frac{1}{2} + \ell_\nu - j_\nu} \begin{pmatrix} -i \frac{\kappa_\nu}{|\kappa_\nu|} j_{\bar{\ell}_\nu} X_{-\kappa_\nu - \mu_\nu} \\ j_{\ell_\nu} X_{\ell_\nu - \mu_\nu} \end{pmatrix}$$

where this has been normalized to

$$\int \psi_{\kappa\mu}(W') \ \psi_{\kappa\mu}(W) \ d\tau = \delta(W - W')$$

so that the density of states equals one

$$\rho_\nu \ (W) = 1$$

The electron is described at large distances as a distorted plane wave of momentum $\vec{p}$ and spin $\vec{\sigma}$, which can be expanded[7] as

$$\psi_e(p\sigma) = \sum_{\kappa\mu} e^{-i\Delta_\kappa} \sqrt{4\pi \ (2\ell + 1)} \ \ (\ell \tfrac{1}{2} j : 0 \sigma\sigma) \ D^{j}_{\mu\sigma} \ \ (Z \rightarrow p) \ \psi_{\kappa\mu}$$

where the phase shift Δ_κ describes the distortion of the plane wave for a spherical potential and is evaluated by Bhalla and Rose[14] for a uniformly charged daughter nucleus.

The spherical wave function used is

$$\psi_{\kappa\mu} = \left(\frac{W}{\pi p}\right)^{\frac{1}{2}} \begin{pmatrix} G_\kappa & X_{\kappa\mu} \\ iF_\kappa & X_{-\kappa\mu} \end{pmatrix}$$

which is normalized in the energy scale

$$\int \psi_{\kappa\mu}{}^{+}(W)\ \psi_{\kappa\mu}(W')\ d\tau = \delta(W - W')$$

so that the density of states if unity $(\rho_e(W) = 1)$. The F and G are normalized to unity in a sphere of unit radius

$$\int_0^1 (F_\kappa{}^2 + G_\kappa{}^2)\ r^2 dr = 1$$

which is the same as the Bhalla and Rose normalization of the F's and G's which they tabulate.

With these expressions for the electron and neutrino wave functions, the A_4 lepton contribution can be rewritten using (reference 5, p. 174)

$$X^+_{\kappa\mu}\ X_{\kappa'\mu'} = \sum_L \delta_{L+\ell+\ell',\mathrm{even}}\ (-)^{\mu+\frac{1}{2}}\ \rho_L(jj')$$

$$(jj'L{:}\mu - \mu'\ M)\ Y_{L-M}$$

$$X^+_{-\kappa\mu}\ X_{-\kappa\mu'} = X^+_{\kappa\mu}\ X_{\kappa'\mu'}$$

where $\rho_J(JJ') = \left(\frac{(2j+1)(2j'+1)}{4\pi\ (2J+1)}\right)^{\frac{1}{2}}\ (jj'J{:}-\tfrac{1}{2}\ \tfrac{1}{2}\ 0)$

and defining the following

$$\begin{aligned}\overline{D(\kappa\ \kappa_\nu)} &= \overline{D(\kappa\ \kappa_\nu|r|)} \\ &= j_{\ell_\nu}(qr)\ G_\kappa(r) - \frac{\kappa_\nu}{|\kappa_\nu|}\ j_{\overline{\ell}_\nu}\ F_\kappa(r)\end{aligned}$$

$$A_4(LL\ \kappa\ -\ \kappa_\nu) = \pm\ \delta_{\ell+\ell_\nu+L,\mathrm{even}}\ \overline{D(\kappa\ \kappa_\nu)}$$

$$+\ i\ \frac{\kappa_\nu}{|\kappa_\nu|}\ \delta_{\ell+\ell_\nu+L,\mathrm{even}}\ \overline{D(\kappa\ -\ \kappa_\nu)}\ .$$

Note that $A_4(L\ \kappa\ -\ \kappa_\nu) = \mp\ i\ \dfrac{\kappa_\nu}{|\kappa_\nu|}\ A_4(LL\ \kappa\ \kappa_\nu)$ for $e^{\mp}$.

Thus after simplification

$$\psi^+_{\kappa\mu}(1 \pm \gamma_5)\ \psi^c_{\kappa_\nu\mu_\nu} = \sum_L (-)^{\mu+\mu_\nu+\frac{1}{2}+\ell_\nu-j}\ \frac{q}{\pi}\left(\frac{W}{p}\right)^{\frac{1}{2}}$$

$$\rho_L(jj_\nu)(jj_\nu L{:}\mu\mu_\nu M)\ Y_{L-M}\ A_4(LL\ \kappa\ \kappa_\nu)$$

With this, the A_4 lepton contribution is

$$A_4 = \sum_{\kappa\mu\kappa_\nu\mu_\nu L} 4q\left(\frac{W}{p}\right)^{\frac{1}{2}} e^{i\Delta_\kappa}\sqrt{(2\ell + 1)(2\ell_\nu + 1)}\ \ D^{j^*}_{\mu\sigma}\ (Z \rightarrow p)$$

$$D^{j_\nu}_{\mu_\nu\sigma_\nu}\ (Z \rightarrow q)(\ell\ \tfrac{1}{2}j{:}0\sigma\sigma)(\ell_\nu\ \tfrac{1}{2}j_\nu{:}0\sigma_\nu\sigma_\nu)\ (-)^{\mu+\mu_\nu\frac{1}{2}-j_\nu+\ell_\nu}\ \rho_L(jj_\nu)$$

$$(jj_\nu L{:}\mu\mu_\nu M)\ Y_{L-M}\ A_4(LL\ \kappa\ \kappa_\nu)$$

Similarly, an expression for the $\vec{A}$ term may be developed using the following relations and definitions (reference 5, p. 177).

$$X^{+}_{\kappa\mu}\ \vec{\sigma}\ X_{\kappa'\mu'} = (-)^{\mu+\frac{1}{2}} \sum_{LJ} \rho_J(jj')(jj'J:\mu - \mu'M)\ \delta_{\ell+\ell'+L,\mathrm{even}}$$

$$\vec{V}^{L}_{J-M} \left\{(J1L:000) + \frac{\kappa}{|\kappa|} w_J(jj')(J1L:1-10)\right\}$$

where $\quad w_0(jj') = 0$

$$w_J(jj') = \frac{2j + 1 + (2j' + 1)(-)^{j+j'+J}}{\sqrt{2J(2J + 1)}} \ ; \ J \neq 0$$

$\vec{V}^{L}_{JM}$ is the "vector spherical harmonic" (reference 5, p. 106)

$$\vec{V}^{L}_{JM} = \sum_{m'=0,-1} (L1J:m - m', m', m)\ Y_{L,m-m'}\ \hat{e}_{m'}$$

where the spherical unit vectors are defined as

$$\hat{e}_0 = \hat{e}_Z$$

$$\hat{e}_{\pm 1} = \frac{\hat{e}_x \pm i\ \hat{e}_y}{\sqrt{2}}$$

Their properties are

$$\hat{e}_m{}^{+} \cdot \hat{e}_m{}' \equiv \delta_{mm'} = (-)^m\ \hat{e}_{-m} \cdot \hat{e}_{m'}$$

$$\hat{e}_m \times \hat{e}_{m'} = i\sqrt{2}\ (111:m\ m'n)\ \hat{e}_n$$

$$= 0 \quad \text{for } m = m'$$

$$= i\ \frac{m - m'}{|m - m'|}\ \hat{e}_{m+m'} \quad \text{for } m \neq m'$$

These "vector spherical harmonics" are related to the irreducible spherical tensors, $T^{M}_{JL}(\hat{r}\ \vec{B})$, of Rose and Osborn.[10]

$$T^{M}_{JL}(\hat{r}\ \vec{B}) = \sum_{m'} (1LJ:-m',\ m' + m,\ m)\ Y_{L,m'+m}\ Y_{1-m'}(\vec{B})$$

where $\quad Y_{1m}(\vec{B}) = |B|\ Y_{1m}(\hat{B})$

and $\quad B_m = \vec{B}\cdot\hat{e}_m = \left(\frac{4\pi}{3}\right)^{\frac{1}{2}} |B| Y_{1m}(\hat{B})$

Therefore $\quad \vec{V}^{L}_{Jm}\cdot\vec{B} = (-)^{J-L-1} \left(\frac{4\pi}{3}\right)^{\frac{1}{2}}\ T^{m}_{JL}(\hat{r}\ \vec{B})$

The definitions introduced are

$$D(\kappa\ \kappa_\nu) = j_{\ell_\nu}\ G_\kappa + \frac{\kappa_\nu}{|\kappa_\nu|}\ j_{\bar{\ell}_\nu}\ F_\kappa$$

$$D_{JL}(\kappa\ \kappa_\nu) = \overline{D(\kappa\ \kappa_\nu)}\ (j1L:000) + \frac{\kappa}{|\kappa_\nu|}\ D(\kappa\ \kappa_\nu)\ w_J(jj_\nu)(J1L:1\text{-}10)$$

$$A(JL\kappa\ \kappa_\nu) = D_{JL}(\kappa\ \kappa_\nu)\ \delta_{\ell+\ell_\nu+L,\text{even}} \pm i\ \frac{\kappa_\nu}{|\kappa_\nu|}\ D_{JL}(\kappa\ -\ \kappa_\nu)\ \delta_{\ell+\bar{\ell}_\nu+L,\text{even}}\ ;$$

for $e^{\mp}$

Note that

$$A(JL\kappa\ -\ \kappa_\nu) = \mp\ i\ \frac{\kappa_\nu}{|\kappa_\nu|}\ A(JL\kappa\ \kappa_\nu) \quad \text{for } e^{\mp}$$

Using the above, the $\vec{A}$ lepton term can be rewritten as

Table 15. The $A_j(JL\kappa\kappa_\nu)$ for the Decay $1^- \rightarrow 2^+$

J	L	κ	κ_ν	$A_4(JL\kappa\kappa_\nu)$
1	1	1	1	$i\ (j_0 G_1 + j_1 F_1)$
1	1	-1	1	$-\ (j_0 F_{-1} - j_1 G_{-1})$
1	1	2	1	$-\ (j_0 F_2 - j_1 G_2)$
1	1	-2	1	$i\ (j_0 G_{-2} + j_1 F_{-2})$
1	1	1	2	$-\ (j_1 F_1 - j_2 G_1)$
1	1	-1	2	$i\ (j_1 G_{-1} + j_2 F_{-1})$
				$A(JL\kappa\kappa_\nu)$
1	0	1	1	$\sqrt{3}\ (j_0 F_1 + \frac{1}{3} j_1 G_1)$
1	0	-1	1	$-\ i\sqrt{3}\ (j_0 G_{-1} - \frac{1}{3} j_1 F_{-1})$
1	0	2	1	$\frac{-i\ 2}{\sqrt{3}}\ j_1 F_2$
1	0	-2	1	$\frac{-i\ 2}{\sqrt{3}}\ j_1 G_{-2}$
1	0	1	2	$\frac{-i\ 2}{\sqrt{3}}\ j_1 G_1$
1	0	-1	2	$\frac{2}{\sqrt{3}}\ j_1 F_{-1}$

Table 15. The $A_j(JL\kappa\kappa_\nu)$ for the Decay $1^- \rightarrow 2^+$ (Concluded)

J	L	κ	κ_ν	$A(JL\kappa\kappa_\nu)$
1	1	1	1	$i\sqrt{2}\,(j_0 G_1 - j_1 F_1)$
1	1	-1	1	$-\sqrt{2}\,(j_0 F_{-1} + j_1 G_{-1})$
1	1	2	1	$\frac{1}{\sqrt{2}}(j_0 F_2 + j_1 G_2)$
1	1	-2	1	$\frac{-i}{\sqrt{2}}(j_0 G_{-2} - j_1 F_{-2})$
1	1	1	2	$\frac{1}{\sqrt{2}}(j_1 F_1 + j_2 G_1)$
1	1	-1	2	$\frac{i}{\sqrt{2}}(j_1 G_1 - j_2 F_{-1})$
2	1	±1	1	0
2	1	2	1	$\frac{5}{\sqrt{10}}(j_0 F_2 + \frac{j_1 G_2}{5})$
2	1	-2	1	$-i\,\frac{5}{\sqrt{10}}(j_0 G_{-2} - \frac{j_1 F_{-2}}{5})$
2	1	1	2	$\frac{-5}{\sqrt{10}}(j_1 F_1 + \frac{j_2 G_1}{5})$
2	1	-1	2	$-i\,\frac{5}{\sqrt{10}}(j_1 G_{-1} - \frac{j_2 F_{-1}}{5})$

$$\vec{A} = \sum_{\kappa\mu\kappa_\nu\mu_\nu LJ} e^{i\Delta_\kappa}\, 4q \left(\frac{W}{p}\right)^{\frac{1}{2}} \sqrt{(2\ell + 1)(2\ell_\nu + 1)}\ \ D^{j*}_{\mu\sigma}\,(Z \to p)\ D^{j}_{\mu_\nu\sigma_\nu}\,(Z \to q)$$

$$(\ell\ \tfrac{1}{2}j{:}0\sigma\sigma)(\ell_\nu\ \tfrac{1}{2}j_\nu{:}0\sigma_\nu\sigma_\nu)(-)^{\mu+\mu_\nu+\frac{1}{2}-j+\ell_\nu}\ \rho_J(jj_\nu)$$

$$(jj_\nu J{:}\mu\mu_\nu M)\ \vec{V}^{L}_{J-M}\ A(JL\kappa\kappa_\nu)$$

Now the beta decay Hamiltonian density can be written as

$$h = \frac{4\ g}{\sqrt{2}}\, q \left(\frac{W}{p}\right)^{\frac{1}{2}} \sum_{\kappa\mu\kappa_\nu\mu_\nu} e^{i\Delta_\kappa}\ (-)^{\mu+\mu_\nu+\frac{1}{2}-j_\nu+\ell_\nu}\ D^{j*}_{\mu\sigma}\,(Z \to p)\ D^{j_\nu}_{\mu_\nu\sigma_\nu}(Z \to q)$$

$$\sqrt{(2\ell + 1)(2\ell_\nu + 1)}\ \ (\ell\ \tfrac{1}{2}j{:}\ 0\sigma\sigma)(\ell_\nu\ \tfrac{1}{2}j_\nu{:}0\sigma_\nu\sigma_\nu)$$

$$\rho_J(jj')(jj'J{:}\mu\mu_\nu M)$$

$$\Big\{A(JL\kappa\kappa_\nu)\ \vec{V}^{L}_{J-M}\cdot U_I^{+}\left(\frac{C_V\vec{p}}{Mc} + C_A\vec{\sigma}\right) U_i$$

$$+\ \delta_{JL}\ A_4(JL\kappa\kappa_\nu)\ Y_{L-M}\ U_I^{+}\left(C_V + C_A\,\frac{\sigma\cdot p}{Mc}\right) U_i$$

$$+\ \delta_{JL}\,\frac{C_A}{2Mc}\,U_I^{+}\ \vec{\sigma}\ U_i\cdot p\ \left\{A_4(JL\kappa\kappa_\nu)\ Y_{L-M}\right\}$$

$$+\ \frac{C_V}{2Mc}\,U_I^{+}\ U_i\ p\cdot\left\{A(JL\kappa\kappa_\nu)\ \vec{V}^{L}_{J-M}\right\}$$

$$+\ i\,\frac{C_V}{2Mc}\,U_I^{+}\ \vec{\sigma}\ U_i\cdot\vec{p}\mathrm{x}\ \left\{A(JL\kappa\kappa_\nu)\ \vec{V}^{L}_{J-M}\right\}\Big\}$$

The first two lines are the terms usually considered. To simplify the last three terms, the following relations will be used, where the D operators are defined as

$$D_{-}(L) = \frac{d}{dr} - \frac{L}{r}$$

$$D_{+}(L) = \frac{d}{dr} + \frac{L+1}{r} = D_{-}(L) + \frac{2L+1}{r}$$

The gradient formula is (reference 25, p. 124)

$$\vec{\nabla}\left\{A_4(LLKK_\nu)\ Y_{L-M}\right\} = -\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}} \vec{V}^{L+1}_{L-M}\ D_{-}(L)\ A_4(LLKK_\nu)$$

$$+\left(\frac{L}{2L+1}\right)^{\frac{1}{2}} \vec{V}^{L-1}_{L-M}\ D_{+}(L)\ A_4(LLKK_\nu)$$

The divergence formula is (reference 25, p. 134)

$$\vec{\nabla}\cdot\left\{A(JLKK_\nu)\ \vec{V}^{L}_{J-M}\right\} = \delta_{JL+1}\left(\frac{L+1}{2L+3}\right)^{\frac{1}{2}} Y_{L+1,-M}\ D_{-}(L)\ A(JLKK_\nu)$$

$$-\ \delta_{JL-1}\left(\frac{L}{2L-1}\right)^{\frac{1}{2}} Y_{L-1,-M}\ D_{+}(L)\ A(JLKK_\nu)$$

The curl formula is

$$\vec{\nabla}\times\left\{A(JLKK_\nu)\ \vec{V}^{L}_{J-M}\right\} =$$

$$\left(\frac{(L+J+3)(L-J+2)(L+J)(J-L+1)}{4\ (2L+1)(2L+3)}\right)^{\frac{1}{2}} V^{L+1}_{J-M}\ D_{-}(L)\ A(JLKK_\nu)$$

$$+ \left(\frac{(L + J + 2)(L + J - 1)(L - J + 1)(J - L + 2)}{4\ (2L + 1)(2L - 1)}\right)^{\frac{1}{2}} V^{L-1}_{J-M}\ D_+(L)\ A(JLKK_\nu)$$

The following definitions are made.

$$D_1(JJKK_\nu) = i^L C_V \left\{A_4(JJKK_\nu)\right.$$

$$- \frac{i\hbar}{2Mc}\left[\delta_{LJ-1}\left(\frac{2L + 1}{2L + 3}\right)^{\frac{1}{2}} D_-(L)\ A(JLKK_\nu)\right.$$

$$\left.\left. + \delta_{LJ+1}\left(\frac{L}{2L - 1}\right)^{\frac{1}{2}} D_+(L)\ A(JLKK_\nu)\right]\right\}$$

$$D_2(JLKK_\nu) = i^L \left\{C_A\ A(JLKK_\nu)\right.$$

$$\frac{i\hbar}{2Mc}\left\{\delta_{LJ+1}\ C_A\left(\frac{J + 1}{2J + 1}\right)^{\frac{1}{2}} D_-(J)\ A_4(JJKK_\nu)\right.$$

$$- \delta_{LJ-1}\ C_A\left(\frac{J}{2J + 1}\right)^{\frac{1}{2}} D_+(J)\ A_4(JJKK_\nu)$$

$$+ \left[C_V\left(\frac{(L+J+2)(L-J+L)(L+J-1)(J-L+2)}{4\ (2L+1)(2L-1)}\right)^{\frac{1}{2}}\right.$$

$$\left. D_-(L-1)\ A(J,L-1,KK_\nu)\right]_{L>0}$$

$$C_V\left(\frac{L+J+3)(L+J)(L-J+2)(J-L+1)}{4\ (2L+1)(2L+3)}\right)^{\frac{1}{2}}$$

$$\left.\left. D_+(L+1)\ A(J,L+1,KK_\nu\right\}\right\}$$

$$D_3(JL\kappa\kappa_\nu) = i^L \frac{C_V}{Mc} A(JL\kappa\kappa_\nu)$$

$$D_4(JJ\kappa\kappa_\nu) = i^L \frac{C_A}{Mc} A_4(JJ\kappa\kappa_\nu)$$

These D's are irreducible spherical tensors. The first term in D_1 and D_2 is the one usually kept, since the factor 1/M reduced the size of the others.

The Hamiltonian density can now be rewritten by performing a multipole expansion on the lepton contribution and be separating the radial lepton contribution $(D_i(JL\kappa\kappa_\nu)$ from the spin and angular dependence for a spherically symmetric potential.

$$h = \frac{g}{\sqrt{2}} q \left(\frac{W}{p}\right)^{\frac{1}{2}} \sum_{\kappa\mu\kappa_\nu\mu_\nu JL} (-i)^L e^{i\Delta_\kappa} (-)^{\mu+\mu_\nu+\frac{1}{2}-j+\ell_\nu} \sqrt{(2\ell + 1)(2\ell_\nu + 1)}$$

$$D^{j*}_{\mu\sigma}(Z \to p)\, D^{j_\nu}_{\mu_\nu\sigma_\nu}(Z \to q)(\ell \tfrac{1}{2}j{:}0\sigma\sigma)(\ell_\nu \tfrac{1}{2}j_\nu{:}0\sigma_\nu\sigma_\nu)(j_\nu J{:}\mu\mu_\nu M)$$

$$\rho_J(jj_\nu) \int \Big\{ D_1(JJ\kappa\kappa_\nu)\, U_I^+\, Y_{J-M}\, U_i + D_2(JL\kappa\kappa_\nu)\, U_I^+\, \sigma\cdot V^L_{J-M}\, U_i$$

$$D_3(JL\kappa\kappa_\nu)\, U_I^+\, V^L_{J-M}\cdot p\, U_i + D_4(JJ\kappa\kappa_\nu)\, U_I^+\, Y_{J-M}\cdot p\, U_i \Big\}\, r^2 dr$$

Table 16. The Lepton Radial Contributions for First Forbidden Beta Decay

$$D_1(11\kappa\kappa_\nu) = i\, C_V A_4(11\kappa\kappa_\nu)$$

$$+ \frac{C_V \hbar}{2Mc\sqrt{3}} \left\{ \frac{d}{dr} A(10\kappa\kappa_\nu) + \sqrt{2}\left(\frac{d}{dr} + \frac{3}{r}\right) A(12\kappa\kappa_\nu) \right\}$$

$$D_2(11\kappa\kappa_\nu) = i\, C_A\, A(11\kappa\kappa_\nu)$$

$$- \frac{\hbar}{2Mc\sqrt{3}} \left\{ C_A \sqrt{2}\left(\frac{d}{dr} - \frac{1}{r}\right) A_4(11\kappa\kappa_\nu) - C_A \left(\frac{d}{dr} + \frac{2}{r}\right) A_4(11\kappa\kappa_\nu) \right.$$

$$\left. + C_V \sqrt{2}\frac{d}{dr} A(10\kappa\kappa_\nu) + C_V \left(\frac{d}{dr} + \frac{3}{r}\right) A(12\kappa\kappa_\nu) \right.$$

$$D_3(10\kappa\kappa_\nu) = \frac{C_V}{Mc} A(10\kappa\kappa_\nu)$$

$$D_2(21\kappa\kappa_\nu) = i\, C_A\, A(21\kappa\kappa_\nu)$$

$$- \frac{\hbar}{2Mc\sqrt{5}} \left\{ C_A \sqrt{3}\left(\frac{d}{dr} - \frac{2}{r}\right) A_4(22\kappa\kappa_\nu) \right.$$

$$\left. - C_A \sqrt{2}\left(\frac{d}{dr} + \frac{3}{r}\right) A_4(22\kappa\kappa_\nu) + C_V \sqrt{3}\left(\frac{d}{dr} + \frac{3}{r}\right) A(22\kappa\kappa_\nu) \right\}$$

APPENDIX III

In this section the expressions for the electron shape and for the beta-gamma angular correlation coefficients will be simplified following the methods of Frauenfelder and Steffen.[7]

For the decay indicated in Figure 1, the probability per unit time that an electron is emitted is given from first order perturbation theory by

$$\lambda = \frac{2\pi}{\hbar} \sum |H_{Ii}|^2 = \frac{2\pi}{\hbar} \int \sum |H_{Ii}(E)|^2 \rho(E)\, dE$$

$$= \int C(E)\, \rho(E)\, dE$$

Here the notation and definitions of Chapter I have been used. Since, in Appendix II, the lepton wave functions have been normalized in the energy scale, the density of final states is unity. Hence the probability of the electron decaying with energy E into the energy range dE per unit time is

$$\frac{d\lambda}{dE} = C(E)$$

The probability, $W(\theta,S)$, that immediately after the electron is emitted, a photon of spin S is emitted at an angle θ with respect to the electron is given by[6,7]

$$W(\theta,S) = \frac{1}{2I_i + 1} \int_\gamma \int_\beta \sum_{m_f m_i} \left| \sum_m H_{Ii}^{\beta} H_{fI}^{\gamma} \right|^2$$

where we have assumed the nuclei are unpolarized and have averaged over the initial states.

In terms of the density matrix introduced in Chapter I, these expressions can be rewritten as

$$W(\theta,S) = \frac{1}{2I_i + 1} \sum_{mm'} \langle m|\rho_\beta|m'\rangle \langle m'|\rho_\gamma|m\rangle$$

$$C(E) = \frac{2\pi}{\hbar} \frac{1}{2I_i + 1} \sum_{mm'} d\Omega p \; \delta_{mm'} \langle m|\rho_\beta|m'\rangle$$

Here $\hbar$ has been set equal to one in rational relativistic units, and the direction of emission of the electron has been integrated over the solid angle for the case when its direction is not observed.

To simplify the density matrix, 3j symbols will be used for their ease of manipulation.

$$(j_1 j_2 j_3 : m_1 m_2 m_3) = (-)^{j_2 - j_1, -m_3} \sqrt{2j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

The nuclear contribution can be simplified by using the Wigner-Eckart theorem

$$\int U_{Im}^{+} O_{JM}^{*} U_{I_i m_i} r^2 dr \, d\Omega$$

(Continued)

$$\equiv \int (Im:O^{*}_{JM}:I_i m_i)\ r^2 dr$$

$$\equiv \sqrt{2I_i + 1}\ (-)^{J-I-m_i} \begin{pmatrix} I & J & I_i \\ m & M & -m_i \end{pmatrix} \int \langle O_J \rangle\ r^2 dr$$

and the definition

$$M_{JL}(\kappa\kappa_\nu) = \int r^2 dr\ \{D_1(JJ\kappa\kappa_\nu)\ \langle Y_J \rangle + D_4(JJ\kappa\kappa_\nu)\ \langle Y_J \sigma\cdot p \rangle$$

$$(-)^{L+J+1}\ [D_2(JL\kappa\kappa_\nu)\ \langle V_J{}^L\cdot\sigma \rangle + D_3(JL\kappa\kappa_\nu)\ \langle V_J{}^L\cdot p \rangle\ ]\}$$

Using the above and the expression for the density matrix in Appendix II, one can write

$$\langle m|\rho_\beta|m' \rangle = \frac{g^2}{2}\ \frac{Wq^2}{p}\ (2I_i + 1) \int d\Omega_\nu \sum_{\substack{\sigma\sigma_\nu m_i \kappa\kappa'\mu\mu' JL \\ MM'\kappa_\nu\kappa_\nu'\mu_\nu\mu_\nu' J'L'}}$$

$$(-)^{\mu+\mu'+\mu_\nu+\mu_\nu'+3-\ell-\ell'-j-j'+J+J'-2I-2m_i}\ e^{i(\Delta_\kappa-\Delta_\kappa')}$$

$$[(2\ell + 1)(2\ell' + 1)(2\ell_\nu + 1)(2\ell_\nu' + 1)(2j + 1)(2j' + 1)(2j_\nu + 1)$$

$$(2j_\nu' + 1)(2J + 1)(2J' + 1)]^{\frac{1}{2}}\ \rho_J(jj_\nu)\ \rho_{J'}(j'j_\nu')$$

$$\begin{pmatrix} \ell & \frac{1}{2} & j \\ 0 & \sigma & -\sigma \end{pmatrix}\begin{pmatrix} \ell' & \frac{1}{2} & j \\ 0 & \sigma & -\sigma \end{pmatrix}\begin{pmatrix} \ell_\nu & \frac{1}{2} & j \\ 0 & \sigma & -\sigma \end{pmatrix}\begin{pmatrix} \ell_\nu' & \frac{1}{2} & j_\nu' \\ 0 & \sigma_\nu & -\sigma_\nu \end{pmatrix}\begin{pmatrix} j & j_\nu & J \\ \mu & \mu_\nu & -M \end{pmatrix}\begin{pmatrix} j' & j_\nu' & J \\ \mu' & -\mu_\nu' & -M' \end{pmatrix}$$

$$\begin{pmatrix} I & J & I_i \\ m & M & -m_i \end{pmatrix}\begin{pmatrix} I & J' & I_i \\ m' & M' & -m_i \end{pmatrix} D^{j*}_{\mu\sigma}\ (Z \to p)\ D^{j'}_{\mu'\sigma}\ (Z \to p)\ D^{j}_{\mu_\nu\sigma_\nu}\ (Z \to q)\ D^{j_\nu'*}_{\mu_\nu\sigma_\nu}\ (Z \to q)$$

$$M_{JL}(\kappa\kappa_\nu)\, M^*_{J'L'}(\kappa'\kappa_\nu')$$

This sum over the 17 indices can be reduced to eight indices if the following relations are used in the sequence given.

Define $\qquad \Delta_{\kappa\kappa'} \equiv \Delta_\kappa - \Delta_{\kappa'}$

Use: $\qquad 2I + 2m_i = \text{even};$

$$\int D^{j^*}_{\mu_\nu\sigma_\nu'}\, D^{j_\nu}_{\mu_\nu\sigma_\nu}\, d\Omega_\nu = \frac{8\pi^2}{2j_\nu + 1}\, \delta_{\mu_\nu\mu_\nu'}\, \delta_{j_\nu j_\nu'}$$

$$D^{j^*}_{\mu\sigma}\, D^{j'}_{\mu'\sigma'} = (-)^{\mu'-\sigma} \sum_k (2k+1) \begin{pmatrix} j & j' & k \\ \mu & -\mu' & a \end{pmatrix} \begin{pmatrix} j & j' & k \\ \sigma & -\sigma & 0 \end{pmatrix} D^k_{a0}$$

$$\begin{pmatrix} \frac{1}{2} & j_\nu & \ell_\nu'' \\ \sigma_\nu & -\sigma_\nu & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & j_\nu & \ell_\nu' \\ \sigma_\nu & -\sigma_\nu & 0 \end{pmatrix} = \frac{\delta_{\ell_\nu \ell_\nu'}}{2\ell_\nu + 1}$$

$$\delta_{\ell_\nu\ell_\nu'}\, \delta_{j_\nu j_\nu'} = \delta_{\kappa_\nu\kappa_\nu'}$$

$$\begin{pmatrix} j & \frac{1}{2} & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} = \frac{(-)^{j-\frac{1}{2}}}{\sqrt{2(2J+1)}}\, \delta_{J,j\pm\frac{1}{2}}$$

$$\begin{pmatrix} j & \frac{1}{2} & J \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \pm\, \frac{(-)^{j-\frac{1}{2}}}{\sqrt{2(2J+1)}}\, \delta_{J,j\pm\frac{1}{2}}$$

$$\sum_{\sigma=\pm\frac{1}{2}} (-)^{-\sigma} \begin{pmatrix} j & \frac{1}{2} & \ell \\ \sigma & -\sigma & 0 \end{pmatrix} \begin{pmatrix} j' & \frac{1}{2} & \ell' \\ \sigma & -\sigma & 0 \end{pmatrix} \begin{pmatrix} j & j' & k \\ \sigma & -\sigma & 0 \end{pmatrix}$$

$$= \frac{\delta_{k+\ell+\ell',\mathrm{even}}}{\sqrt{(2\ell+1)(2\ell'+1)}} \; \delta_{\ell,j\pm\frac{1}{2}} \, \delta_{\ell',j'\pm\frac{1}{2}} \begin{pmatrix} j & j' & k \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} (-)^{j+j'-\frac{3}{2}}$$

$$\sum_{\mu\mu_\nu M} (-)^{\mu+2\mu'} \begin{pmatrix} J & j & j_\nu \\ -M & \mu & \mu_\nu \end{pmatrix} \begin{pmatrix} j' & J' & j_\nu \\ -\mu' & M' & -\mu_\nu \end{pmatrix} \begin{pmatrix} j' & j & k \\ \mu' & -\mu & -a \end{pmatrix}$$

$$= (-)^{M'-j'-j-j_\nu} \begin{pmatrix} J & J' & k \\ -M & M' & -a \end{pmatrix} \begin{Bmatrix} J & J' & k \\ j' & j & j_\nu \end{Bmatrix}$$

$$\sum_{m_i MM'} (-)^{M'+J+J'+k+M+m_i-m} \begin{pmatrix} I & J' & I_i \\ m' & M' & -m_i \end{pmatrix} \begin{pmatrix} J & I & I_i \\ -M & -m & m_i \end{pmatrix} \begin{pmatrix} J & J' & k \\ M & -M' & a \end{pmatrix}$$

$$= (-)^{k-I_i-m} \begin{pmatrix} I & I & k \\ m' & -m & a \end{pmatrix} \begin{Bmatrix} I & I & k \\ J & J' & I_i \end{Bmatrix}$$

Finally defining

$$d_k(JJ') = (2\pi g)^2 \sum_{\kappa\kappa'\kappa_\nu LL'} \delta_{\ell+\ell'+k,\mathrm{even}} \, e^{i\Delta_{\kappa\kappa'}} \, \rho_J(jj_\nu) \, \rho_{J'}(j'j_\nu)$$

$$[(2j+1)(2j'+1)(2J+1)(2J'+1)]^{\frac{1}{2}} \, (-)^{j'+j+j_\nu-\frac{1}{2}}$$

$$\begin{pmatrix} j & j' & k \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \begin{Bmatrix} J & J' & k \\ j' & j & j_\nu \end{Bmatrix} M_{JL}(\kappa\kappa_\nu) \, M^*_{J'L'}(\kappa'\kappa_\nu)$$

then the density matrix can be written as

$$\langle m|\rho_\beta|m'\rangle = \frac{Wq^2}{p} \sum_{kJJ'} (-)^{2I-I_i+m+J+J'} \frac{2k+1}{2I_i+1}$$

$$\begin{pmatrix} I & I & k \\ m' & -m & a \end{pmatrix} \begin{Bmatrix} I & I & k \\ J & J' & I_i \end{Bmatrix} A^k_{a0} \ (Z \to p) \ d_k(JJ')$$

Note the 6-j symbol gives the condition $|I-I_i| \leq J, J' \leq I + I_i$

Next, to simplify the expression for the shape, use

$$\int D^k_{a0} \, d\Omega_p = \frac{4\pi}{\sqrt{2k+1}} \, \delta_{k0}$$

$$\begin{pmatrix} j & j' & 0 \\ m & m' & 0 \end{pmatrix} = \frac{(-)^{j-m}}{\sqrt{2j+1}} \delta_{k0}$$

$$\begin{Bmatrix} j & J' & 0 \\ j' & j & j_\nu \end{Bmatrix} = \frac{(-)^{J+j+j_\nu}}{\sqrt{(2J+1)(2j+1)}} \, \delta_{JJ'} \, \delta_{jj'}$$

and $\sum_m 1 = 2I + 1$.

Then

$$C \equiv 2^6 \, \pi^4 \, g^2 \, \frac{Wq^2}{p} \sum_{\kappa\kappa_\nu J} \Big| \sum_L \rho_J(jj_\nu) \; M_{JL}(\kappa\kappa_\nu) \Big|^2$$

$$= 2^3 \, \pi^2 \, \frac{Wq^2}{p} \sum_J \frac{(-)^J}{\sqrt{2J+1}} \, d_0(JJ)$$

To get an expression for angular correlation coefficients, we use the gamma decay density matrix (reference 8', p. 1022)

$$\langle m' | \rho_\gamma | m \rangle = \sum_{LL_\gamma' kNa} (-)^{k-I_f-m-L_\gamma'} \sqrt{2k+1} \, C^*_{ka} \, (L_\gamma L_\gamma')$$

$$\begin{pmatrix} I & I & k \\ m' & -m & N \end{pmatrix} \begin{Bmatrix} I & I & k_2 \\ L_\gamma & L_\gamma' & I_f \end{Bmatrix} \langle I_f \| L_\gamma \| I \rangle \langle I_f \| L_\gamma' \| I \rangle^* \; D^{k^*}_{Na} \; (Z \to \gamma)$$

where

$$C_{k0}(L_\gamma L_\gamma'; S) = S^k \; (-)^{L_\gamma - 1} \sqrt{(2k+1)(2L_\gamma + 1)(2L_\gamma' + 1)} \begin{pmatrix} L_\gamma & L_\gamma' & k \\ 1 & -1 & 0 \end{pmatrix}$$

Using the above and the following relations, the beta-gamma angular correlation function can be simplified.

$$\sum_{mm'} \begin{pmatrix} I & I & k \\ m' & -m & a \end{pmatrix} \begin{pmatrix} I & I & k \\ m' & -m & N_2 \end{pmatrix} = \frac{\delta_{kk_2} \; \delta_{aN}}{2k+1}$$

$$\sum_N D^k_{N0} \; (Z \to p) \; D^{k^*}_{N0} \; (Z \to \gamma) = D^k_{00} \; (\gamma \to p)$$

$$= P_k \; (\cos \theta_{\gamma\beta})$$

$$W(\beta\gamma \; S) = \frac{1}{2I_i + 1} \sum_{mm'} \langle m | \rho_\beta | m' \rangle \langle m' | \rho_\gamma | m \rangle$$

$$= \sum_k S^k \; A_k' \; P_k \; (\cos \theta)$$

$$\frac{W(\beta\gamma \; S)}{A_0'} = 1 + \sum_{k=1} S^k \; A_k \; P_k \; (\cos \theta)$$

$$A_k' = \frac{Wq^2}{p} \; (-)^{I - I_i - 2I_f} \; S^k \sqrt{2k+1} \sum_{JJ'} (-)^{J+J'} \begin{Bmatrix} I & I & k \\ J & J' & I_i \end{Bmatrix} d_k(JJ')$$

$$\sum_{LL'} (-)^{L_\gamma - L_\gamma'} \; F_k(L_\gamma L_\gamma' I_f I) \langle I_f \| L_\gamma \| I \rangle \langle I_f \| L_\gamma' \| J \rangle^*$$

The A_0' term is related to the shape by

$$A_0' = \frac{1}{4\pi} \frac{C(E)}{2\pi} \sum_{L_\gamma} \frac{|\langle I_f \| L_\gamma \| I \rangle|^2}{2I + 1}$$

Hence the beta-gamma angular correlation coefficient can be rewritten as

$$A_k = A_0' = 8\pi^2 \frac{Wq^2}{p} \frac{\sqrt{2I + 1}}{C(E)} (-)^{I-I_i-2I_f} \sqrt{2k + 1}\, (-)^k$$

$$\sum_{JJ'} (-)^{J+J'} \begin{Bmatrix} I & I & k \\ J & J' & I_i \end{Bmatrix} d_k(JJ')\, A_k(\gamma)$$

where

$$A_k(\gamma) = \sum_{L_\gamma L_\gamma'} \frac{(-)^{L_\gamma - L_\gamma'} F_k(L_\gamma L_\gamma' I_f I) \langle I_f \| L_\gamma \| I \rangle \langle I_f \| L_\gamma' \| I \rangle^*}{\sum\limits_{L_\gamma} |\langle I_f \| L_\gamma \| I \rangle|^2}$$

For pure multipole radiation $(L_\gamma = L_\gamma')$ $A_k(\gamma)$ reduces to $F_k(L_\gamma L_\gamma I_f I)$.

In summary, after performing some tedious Racah algebra, the expressions for the shape and beta-gamma angular correlation coefficients have been simplified. Note that A_0 is proportional to the shape.

APPENDIX IV

In this section, the explicit expressions for a $1^-(\beta)\ 2^+(\gamma)\ 0^+$ decay will be written and compared with expressions given by Morita and Morita.[4] For this beta decay the 6-j symbol gives

$$|I_i - I| = 1 \leq J, J' \leq 3 = I_i + I$$

and the contribution due to the pure electric quadrupole transition to the $A_2(\beta,\gamma)$ coefficient is (reference 7, p. 1197)

$$A_2(\gamma) = F_2(2202) = -\left(\frac{5}{14}\right)^{\frac{1}{2}}$$

Putting this into the expressions developed in Appendix III, the shape and A_2 coefficient reduce to

$$C = 2^5\ \pi^4\ g^2\ \frac{Wq^2}{p} \sum_{J=1}^{3} \sum_{\kappa\kappa_\nu} \left| \sum_{L} \rho_J(jj_\nu)\ M_{JL}(\kappa\kappa_\nu) \right|^2$$

$$A_2(\beta\gamma) = \frac{5\left(\frac{5}{14}\right)^{\frac{1}{2}}}{\sum\limits_{J=1}^{3} \sum\limits_{\kappa\kappa_\nu} \left| \sum\limits_{L} \rho_J(jj_\nu)\ M_{JL}(\kappa\kappa_\nu) \right|^2} \sum_{J,J'=1}^{3} \sum_{\kappa\kappa'\kappa_\nu}$$

$$\delta_{\ell+\ell',\mathrm{even}}\ (-)^{J+J'+j+j+j_\nu-\frac{1}{2}}\ e^{i\Delta_{\kappa\kappa'}}$$

$$\left[(2j + 1)(2j' + 1)(2J + 1)(2J' + 1)\right]^{\frac{1}{2}} \begin{Bmatrix} 2 & 2 & 2 \\ J & J' & 1 \end{Bmatrix} \begin{pmatrix} j & j' & 2 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \begin{Bmatrix} J & J' & 2 \\ j' & j & j_\nu \end{Bmatrix}$$

$$\sum_L \rho_J(jj_\nu)\, M_{JL}(\kappa\kappa_\nu) \left[\sum_L \rho_{J'}(j'j_\nu)\, M^*_{J'L'}(\kappa'\kappa_\nu) \right]^*$$

Where now the M_{JL} contains only the reduced nuclear matrix elements with the odd parity operators since there is a change in parity of the initial and intermediate states. They are

$$\langle Y_1\rangle\ ,\ \langle V_1{}^1\cdot\sigma\rangle\ ,\ \langle V_1{}^0\cdot p\rangle\ ,\ \langle V_1{}^2\cdot p\rangle\ ,$$

$$\langle V_2{}^1\cdot\sigma\rangle\ ,\ \langle V_2{}^3\cdot\sigma\rangle\ ,\ \langle V_2{}^2\cdot p\rangle\ ,\ \langle Y_2\,\sigma\cdot p\rangle\ ,$$

$$\langle Y_3\rangle\ ,\ \langle V_3{}^3\cdot\sigma\rangle\ ,\ \langle V_3{}^2\cdot p\rangle,\ \text{and}\ \langle V_3{}^4\cdot p\rangle\ .$$

Next we will make the following approximations which will reduce our expressions to those of Morita and Morita.

Neglect the small radial lepton contributions

$$\sum_{\substack{\kappa=\pm1,\pm2,\pm3\ \ldots \\ \kappa_\nu=\pm1,\pm2,\pm3\ \ldots \\ \kappa'=\pm1,\pm2,\pm3\ \ldots}} \simeq \sum_{\substack{\kappa=\pm1,\pm2. \\ \kappa_\nu=\pm1,\pm2. \\ \kappa_\nu'=\pm1,\pm2.}}$$

Also neglect the terms

$$j_1G_2,\ j_1F_{-2},\ j_2G_1,\ j_2F_{-1},\ j_1G_1,\ j_1F_{-1},\ j_1F_2\ ,\ \text{and}\ j_1G_{-2}$$

which always appear added to a larger lepton contribution (see Table 15).

Also keep only the first term in the D's.

$$D_1(JJ\kappa\kappa_\nu) \simeq i^J\, C_V\, A_4(JJ\kappa\kappa_\nu)$$

$$D_2(JL\kappa\kappa_\nu) \simeq i^L\, C_A\, A(JL\kappa\kappa_\nu)$$

Then keeping only the four reduced nuclear matrix elements $\langle Y_1 \rangle$, $\langle V_1{}^1 \cdot \sigma \rangle$, $\langle V_1{}^0 \cdot p \rangle$, and $\langle V_2{}^1 \cdot \sigma \rangle$, the M_{JL} terms are

$$\sum_L M_{1L}(\kappa\kappa_\nu) \simeq \int r^2 dr \left\{ i\, C_V\, A_4(11\kappa\kappa_\nu) \langle Y_1 \rangle - i\, C_A\, A(11\kappa\kappa_\nu) \langle V_1{}^1 \cdot \sigma \rangle + \frac{C_V}{Mc} A(10\kappa\kappa_\nu) \langle V_1{}^0 \cdot p \rangle \right\}$$

$$\sum_L M_{2L}(\kappa\kappa_\nu) \simeq \int r^2 dr\; i\, C_A\, A(21\kappa\kappa_\nu) \langle V_2{}^1 \cdot \sigma \rangle$$

and

$$\sum_L M_{3L}(\kappa\kappa_\nu) \simeq 0$$

Using these approximations, the expression for the shape has been written for the above decay in Table 17. Making the further approximation

$$j_\ell(qr) \simeq \frac{(qr)^\ell}{(2\ell + 1)!!}$$

for small momenta and making the radial approximation

$$\int A_i(JLKK_\nu) <O_J{}^L> r^2 dr \simeq \left[\frac{A_i(JLKK_\nu)}{r^L}\right]_{R_{NS}} \int <O_J{}^L> r^{2+L} dr$$

and using the relations between the Morita and Morita nuclear matrix elements and ours (given in Table 1), we can now compare this with the Morita and Morita[4] shape factor

$$C_{MM} = -\frac{b_{11}{}^{0}}{\sqrt{3}} + \frac{b_{22}{}^{0}}{\sqrt{5}}$$

Doing this we get

$$C(W) = \frac{d\lambda}{dW} = \frac{5}{3}(2\pi g)^2 \, C_{MM} \, F(Z,W) \, Wq^2 \, p$$

Hence Morita and Morita's shape factor is proportional to the one defined in Chapter I.

Doing the same thing for the A_2 coefficient, which has been rewritten in Table 18, we get that our A_2 coefficient is identical to Morita and Morita's

$$A_2 = A_2(MM) = \frac{1}{C_{MM}}\left[\frac{b_{11}{}^{2} - b_{12}{}^{2}}{2\sqrt{6}} - \frac{b_{22}{}^{2}}{2\sqrt{14}}\right]$$

Table 17. The Shape for a $1^-(\beta)\ 2^+$ Decay

$$C = (2\pi)^3\ g^2\ \frac{Wq^2}{p}\ \frac{2}{3}\ C_V{}^2$$

$$\Big\{\Big| \int r^2 dr\ \{-\ (j_0 G_1 + j_1 F_1)\ \langle Y_1\rangle\ + \sqrt{2}\,\frac{C_A}{C_V}\ (j_0 G_1 - j_1 F_1)\ \langle V_1{}^1\cdot\sigma\rangle$$

$$+ \frac{\sqrt{3}}{Mc}\ j\ F\ \ \langle V\ \ \cdot p\rangle\ \}\Big|^2$$

$$+\Big| \int r^2 dr\ \{-\ i(j_0 F_{-1} - j_1 G_{-1})\ \langle Y_1\rangle + i\ \sqrt{2}\,\frac{C_A}{C_V}\ (j_0 F_{-1} + j_1 G_{-1})\ \langle V_1{}^1\cdot\sigma\rangle$$

$$-\ i\ \frac{\sqrt{3}}{Mc}\ j\ G_-\ \ \langle V\ \ \cdot p\rangle\ \}\Big|^2$$

$$+\ 2\Big| \int r^2 dr\ \{-\ i\ j_0 F_2\ \langle Y_1\rangle\ - i\ \frac{C_A}{C_V}\ \frac{j_0 F_2}{\sqrt{2}}\ \langle V_1{}^1\cdot\sigma\rangle\ \}\Big|^2$$

$$+\ 2\Big| \int r^2 dr\ \{-\ j_0 G_{-2}\ \langle Y_1\rangle\ -\ \frac{C_A}{C_V}\ \frac{j_0 G_2}{\sqrt{2}}\ \langle V_1{}^1\cdot\sigma\rangle\ \}\Big|^2$$

$$+\ 2\Big| \int r^2 dr\ \{-\ i\ j_1 F_1\ \langle Y_1\rangle\ -\ i\ \frac{C_A}{C_V}\ \frac{j_1 F_1}{\sqrt{2}}\ \langle V_1{}^1\cdot\sigma\rangle\ \}\Big|^2$$

$$+\ 2\Big| \int r^2 dr\ \{-\ j_1 G_{-1}\ \langle Y_1\rangle\ +\ \frac{C_A}{C_V}\ \frac{j_1 G_1}{\sqrt{2}}\ \langle V_1{}^1\cdot\sigma\rangle\ \}\Big|^2$$

$$+\ 3\Big| \int r^2 dr\ i\ \frac{C_A}{C_V}\ j_0 F_2\ \langle V_2{}^1\cdot\sigma\rangle\ \Big|^2 +\ 3\Big| \int r^2 dr\ \frac{C_A}{C_V}\ j_0 G_{-2}\ \langle V_2{}^1\cdot\sigma\rangle\ \Big|^2$$

$$+\ 3\Big| - \int r^2 dr\ i\ \frac{C_A}{C_V}\ j_1 F_1\ \langle V_2{}^1\cdot\sigma\rangle\ \Big|^2 +\ 3\Big| \int r^2 dr\ \frac{C_A}{C_V}\ j_1 G_{-1}\ \langle V_2{}^1\cdot\sigma\rangle\ \Big|^2\Big\}$$

Table 18. The A_2 Coefficient for a $1^-(\beta)\ 2^+(\gamma)\ 0^+$ Decay

$$A_2(\beta,\gamma) = \frac{(2\pi)^3\ g^2\ \frac{Wq^2}{p}}{C(W)}\ \frac{2}{3} \sum_{LL'}$$

$$\Big\{2\ [\cos \Delta_{1-2}\ M_{1L}(11)\ M^*_{1L'}\ (-21) + \cos \Delta_{-12}\ M_{1L}(11)\ M^*_{1L'}\ (21)$$

$$+ |M_{1L}(21)|^2 + |M_{1L}(-21)|^2$$

$$- 2\left(\frac{3}{5}\right)^{\frac{1}{2}} [\cos \Delta_{2-1}\ M_{2L}(21)\ M^*_{1L'}\ (21) + \cos \Delta_{-21}\ M_{2L}(-21)\ M^*_{1L'}\ (11)$$

$$- M_{1L}(21)\ M^*_{2L'}\ (21) - M_{1L}(-21)\ M^*_{2L'}(-21)$$

$$+ \frac{3}{5}\ [\ |M_{2L}(21)|^2 + |M_{2L}(-21)|^2]\Big\}$$

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*Abbreviations used here follow the form employed in Science Abstracts, Section A Physics Abstracts.

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